The Logic of Provability

Suggestions for exercises. Week 3.

1. (B) (C)

   We have defined $\exists x < y A(x)$ to be short for $\exists x(x < y \wedge A(x))$ and dually $\forall x < y A(x)$ to be short for $\forall x(x < y \to A(x))$. Show that $\exists x < y A(x) \leftrightarrow \neg \forall x < y \neg A(x)$ is provable in PA. Do the same for the dual statement.

2. (C)

   Give a formula $A(x, y)$ that is to hold if and only if x divides y. Show that we can chose $A(x, y)$ to be a Δ_0 formula. We will write $x \mid y$ from now on.

3. (C)

   Give a Δ_0 formula $\text{Prime}(x)$ that is to hold on all prime numbers and on no other numbers.

4. (C)

   Give a Π_1 sentence that expresses Goldbach's conjecture.

5. (C) (A)

   Carry out the full proof that all atomic formulas are Σ_1 formulas as outlined on page 25.

6. (C)

   Here we denote both the number zero and its numeral by 0. Write down both $\overline{\text{GN}(0)}$ and $\text{GN}(\overline{0})$.

7. (C)

   Show that $\mathsf{PA} \vdash v_0 = 16 \to \text{GN}(v_0) = 17$.

8. (D)(*)

   By the Church-Turing thesis we know that some Σ_1 formula $\pi(x, y)$ holds for (and only for) pairs of numbers (n, m) such that m is the nth prime number. Sketch this formula. You may use previously defined formulas and formulas defined in the book. (For clarity: we would thus like to have $\pi(1, 2)$, $\pi(2, 3)$, $\pi(3, 5)$, $\pi(4, 7)$, $\pi(5, 11)$ and so on.)

9. (C)

   Do we have $\vdash \mathrm{Bew}(\ulcorner\varphi\urcorner) \to \mathrm{Bew}[\varphi]$? And do we have $\vdash \mathrm{Bew}[\varphi] \to \mathrm{Bew}(\ulcorner\varphi\urcorner)$? Provide a proof or a counterexample.

10. (C)

    Prove that $\vdash u \cdot v = w \to \mathrm{Bew}[u \cdot v = w]$ in the same style as this was done for addition on page 47.

11. (D)

    Prove in PA that $\exists x \exists y(ax + 1 = by) \to \exists x' \exists y'(bx' + 1 = ay')$ whenever $b > 1$. (Hint: multiply by $b - 1$.) Use this to give a simplified proof of theorem (30) of chapter 2 of the book by employing the notion "i is good enough for a and b" being $\exists x \exists y(ax + i = by) \vee \exists x \exists y(bx + i = ay)$.

12. (C) (D)

    Provide proofs for (18)-(22) on page 27.

13. (D)

    Prove that every formula φ in predicate logic is equivalent to one in prenex normal form.

14. (D)

    Give various disabbreviations of $B(h(g(\overline{x})), g(\overline{x}))$ where h and g denote Σ-pterms (h is unary!) and B denotes some definable binary predicate.

15. (C)

    Give Σ-pterms such that

    - $h(0) = 1 \quad h(1) = 2 \quad h(2) = 3 \quad h(3) = 4 \quad h(4) = 5$
    - $h(0) = 9 \quad h(1) = 16 \quad h(2) = 25 \quad h(3) = 36 \quad h(4) = 49$
    - $h(0) = 5 \quad h(1) = 7 \quad h(2) = 11 \quad h(3) = 13 \quad h(4) = 17$

16. (C)

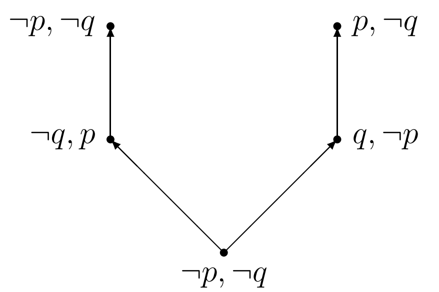
    Show that
    $\mathbf{T} \vdash \Box A \to \Diamond A$ and
    $\mathbf{K4} \vdash \Box A \to (\Box\Box(A \to B) \to \Box\Box B)$.

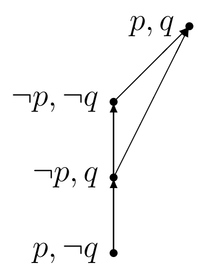
17. (C)

Show that $\mathbf{GL} \vdash \Box(\Box^m\bot \to \Box^n\bot) \leftrightarrow \Box^{n+1}\bot$ whenever $m > n \geq 0$.

18. (C)

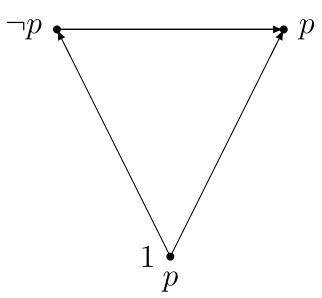
Determine if the following formulas are valid in the lowermost worlds of the two Kripke models below: $\Box p$, $\Box q$, $\Box p \land q$, $\Box\Box\bot$, $\Diamond(q \land \Diamond(p \land \neg q))$.

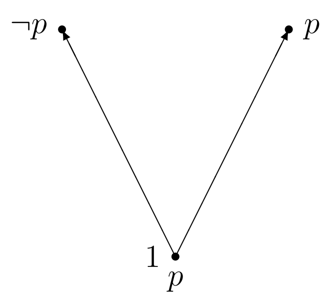




19. (C)

Find a formula which is true in the world 1 of the first model, but not in 1 of the second model.





20. (C)

Determine which of the following formulas are derivable in $\mathbf{K}$:

(a) $\Box\Box p \to \Box p$

(b) $\Box p \land \neg\Box\bot \to \neg\Box\neg p$

(c) $\Box p \land \Box\Box(p \to q) \to \Box\Box q$

(d) $\Box\Box p \wedge \Box\Box(p \to q) \to \Box\Box q$

Give proofs of derivable formulas and Kripke countermodels for the nonderivable ones.

21. (D)

(a) How many Kripke frames are there on a single element set? Depict them all. (b) The same question for a two-element set.

22. (D)

Assume a Kripke frame[1] has n elements and the language has m propositional variables. How many different Kripke models exist on this frame?

23. (C) (A)

Prove the following facts by constructing appropriate Kripke countermodels:

(a) $\mathbf{K4} \nvdash \neg p$;

(b) $\mathbf{K4} \nvdash \Box(\Box p \to p) \to \Box p$;

(c) $\mathbf{K4} \nvdash \neg(\Box p \to p)$;

(d) $\mathbf{S4} \nvdash \Box(\Box(p \to \Box p) \to p) \to p$;
(Hint: a model with just 2 nodes is sufficient.)

24. (D)

Show that the formula $\Box(\Box p \to q) \vee \Box(\Box q \to p)$ is valid in all linearly ordered Kripke models (more generally, if the relation R is reflexive and linear).

25. (C) (A)

How many pairwise inequivalent formulas in one propositional variable are there (a) in classical propositional logic; (b) in **K**4.
(Answer for (b): infinitely many. Hint: iterate $\Box$. Show inequivalence by exhibiting countermodels.)

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[1] Recall: a model is a frame together with a truth assignment of propositional variables.