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Comparing IΣ_1 and PRA

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Abstract. In this paper we will state and prove some comparative theorems concerning PRA and IΣ_1. We shall provide a characterization of IΣ_1 in terms of PRA and iterations of a class of functions.

We will formulate a sufficient condition for a model of PRA to be a model of IΣ_1. This condition is used to give a model-theoretic proof of Parsons' theorem, that is, IΣ_1 is Π_2-conservative over PRA. We shall also give a purely syntactical proof of Parsons' theorem.

Finally, we show that IΣ_1 proves the consistency of PRA on a definable IΣ_1-cut. This implies that proofs in IΣ_1 can have non-elementary speed up over proofs in PRA.

1. Parsons' theorem

Parsons' theorem says that IΣ_1 is Π_2-conservative over PRA. It was proved independently by C. Parsons ([Par70], [Par72]), G. Mints ([Min72]) and G. Takeuti ([Tak75]). Often, PRA is associated with finitism ([Sko67], [HB68], [Tai81]). In this light, Parsons' theorem can be considered of great importance as a partial realization of Hilbert's programme.

The first proofs of Parsons' theorem were all of proof-theoretical nature. Parsons' first proof, [Par70], is based upon Gödel's Dialectica interpretation. His second proof, [Par72], merely relies on a cut-elimination. Mints' proof, [Min72], employs the no-counterexample interpretation of a special sequent calculus. The proof by Takeuti, [Tak75], employs an ordinal analysis in the style of Gentzen.

Over the years, many more proofs of Parsons' theorem have been published. In many accounts Herbrand's theorem plays a central role in providing primitive recursive Skolem functions for Π_2-statements provable in IΣ_1. (Cf. Sieg's method of Herbrand analysis [Sie91], Avigad's proof by his notion of Herbrand saturated models [Avi02], Buss' proof by means of his witness predicates [Bus98], and Ferreira's proof using Herbrand's theorem for Σ_3 and Σ_1-formulas [Fer02].) A first model-theoretic proof is due to Paris and

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Kirby. They employ semi-regular cuts in their proof (cf. [Sim99]:373-381).

This paper will add two more proofs to the long list. The first proof is given in Section 3. It is a proof-theoretic proof and can be seen as a modern version of Parsons' second proof. The main ingredient is the Cut-elimination theorem for Tait's sequent calculus.

The second proof is given in Section 4. It is a model-theoretic proof. A central ingredient is an analysis of the difference between PRA and $\mathrm{I}\Sigma_1$ in terms of iteration of total functions.

2. Primitive recursive arithmetic

Primitive recursive arithmetic, PRA for short, was first introduced by Skolem in [Sko67]. Throughout literature there exist many different variants of PRA. In a sense though, they are all the same, as they are easily seen to be equi-interpretable in a faithful way. In this paper we shall consider theories modulo faithful interpretability.

Reading convention All statements about PRA and other theories in this paper will refer to the definition given in that section.

Often one defines PRA in a language that contains for every primitive recursive function a function symbol plus its defining axioms. In this extended language PRA allows for induction over open formulas. In this section we shall with PRA refer to this theory.

Definition 1 ($\mathrm{I}\Sigma_1^R$). *$\mathrm{I}\Sigma_1^R$ is the predicate logical theory in the pure language of arithmetic $\{+, \cdot, 0, 1, <\}$ that contains Robinson's arithmetic Q plus the Σ_1-induction rule. The Σ_1-induction rule allows one to conclude $\forall x\ \varphi(x, \mathbf{y})$ from $\varphi(0, \mathbf{y}) \wedge \forall x\ (\varphi(x, \mathbf{y}) \rightarrow \varphi(x+1, \mathbf{y}))$.*

It is well known that PRA is faithfully interpretable in $\mathrm{I}\Sigma_1^R$ in the expected way, that is, every function symbol is replaced by its definition in terms of sequences. For a comparison the other way around, we have the following lemma.

Lemma 1. $\mathrm{I}\Sigma_1^R \subseteq \mathrm{PRA}$.

Proof. The proof goes by induction on the length of a proof in $\mathrm{I}\Sigma_1^R$. If $\mathrm{I}\Sigma_1^R \vdash \varphi$ without any applications of the Σ_1 induction rule, it is clear that $\mathrm{PRA} \vdash \varphi$.

So, suppose that the last step in the $\mathrm{I}\Sigma_1^R$-proof of φ were an application of the Σ_1-induction rule. Thus φ is of the form $\forall x\, \exists y\ \varphi_0(x, y, \mathbf{z})$ and we obtain shorter $\mathrm{I}\Sigma_1^R$-proofs of the Σ_1-statements $\exists y\ \varphi_0(0, y, \mathbf{z})$ and $\exists y'\ (\varphi_0(x, y, \mathbf{z}) \rightarrow \varphi_0(x+1, y', \mathbf{z}))$. The induction hypothesis tells us that these statements are also provable in PRA. Herbrand's theorem for PRA provides us with primitive recursive functions $g(\mathbf{z})$ and $h(x, y, \mathbf{z})$ such that

$$\mathrm{PRA} \vdash \varphi_0(0, g(\mathbf{z}), \mathbf{z}) \qquad (1)$$

and

$$\text{PRA} \vdash \varphi_0(x, y, \mathbf{z}) \to \varphi_0(x+1, h(x, y, \mathbf{z}), \mathbf{z}) \qquad (2)$$

Let $f(x, \mathbf{z})$ be the primitive recursive function defined by

$$\begin{cases} f(0, \mathbf{z}) = g(\mathbf{z}), \\ f(x+1, \mathbf{z}) = h(x, f(x, \mathbf{z}), \mathbf{z}). \end{cases}$$

By (1) and (2) it follows from Δ_0-induction in PRA that PRA $\vdash \forall x\ \varphi_0(x, f(x, \mathbf{z}), \mathbf{z})$ whence PRA $\vdash \forall x \exists y\ \varphi_0(x, y, \mathbf{z})$.

3. A proof-theoretic proof of Parsons' theorem

The first proof we give of Parsons' theorem is proof-theoretic. Our presentation is due to L. Beklemishev. It will become evident that the whole argument is easily formalizable as soon as the superexponential function is provably total. This is because our proof only uses the standard Cut-elimination theorem.

In this section we will work with a fragment of first order predicate logic that only contains $\wedge, \vee, \forall, \exists$ and $\neg$, where $\neg$ may only occur on the level of atomic formulae. We can define $\to$ and unrestricted negation as usual. We shall thus freely use these connectives too.

A proof system for this fragment of logic in the form of a Tait calculus is provided in [Sch77]. We will use this calculus in our proof. The calculus works with sequents which are finite sets and should be read disjunctively in the sense that $\Gamma = \{\varphi_1, \ldots, \varphi_n\}$ stands for $\varphi_1 \vee \ldots \vee \varphi_n$. We will omit the set-brackets $\{\}$. The axioms of the Tait calculus are:

$$\Gamma, \varphi, \neg\varphi \qquad \text{for atomic } \varphi.$$

The rules are:

$$\frac{\Gamma, \varphi \quad \Gamma, \psi}{\Gamma, \varphi \wedge \psi}, \qquad \frac{\Gamma, \varphi}{\Gamma, \varphi \vee \psi}, \qquad \frac{\Gamma, \psi}{\Gamma, \varphi \vee \psi},$$

$$\frac{\Gamma, \varphi(a)}{\Gamma, \forall x\ \varphi(x)}, \qquad \frac{\Gamma, \varphi(t)}{\Gamma, \exists x\ \varphi(x)},$$

plus the cut rule

$$\frac{\Gamma, \varphi \quad \Gamma, \neg\varphi}{\Gamma}.$$

In the rule for the universal quantifier introduction it is necessary that the a does not occur free anywhere else in Γ. And in the rule for the introduction of the existential quantifier one requires t to be substitutable for x in φ. In our proof we use the nice properties that this calculus is known to posses. Most notably the cut elimination theorem and some inversion properties.

Let us now fix our versions of PRA and IΣ_1 for this section.

Definition 2 ($\mathrm{I}\Sigma_1$). *The theory $\mathrm{I}\Sigma_1$ is an extension of predicate logic with some easy Π_1-fragment of arithmetic (for example the Π_1-part of Robinson's arithmetic Q), together with all axioms of the form*

$$\forall x\ (\neg \mathsf{Progr}(\varphi, x) \vee \varphi(x, \mathbf{y})).$$

Here φ is some Σ_1-formula and $\mathsf{Progr}(\varphi, x)$ is the Π_2-formula that is equivalent to

$$\varphi(0, \mathbf{y}) \wedge \forall x\ (\varphi(x, \mathbf{y}) \rightarrow \varphi(x+1, \mathbf{y})).$$

Definition 3 (PRA). *The theory $\mathrm{I}\Sigma_1^R$, also called Primitive Recursive Arithmetic, is the extension of predicate logic that arises by adding a simple Π_1-fragment of arithmetic*[1] *together with the Σ_1-induction rule to it. Here, the Σ_1-induction rule is*

$$\frac{\Gamma, \varphi(0, \mathbf{y}) \qquad \Gamma, \forall x\ (\neg\varphi(x, \mathbf{y}) \vee \varphi(x+1, \mathbf{y}))}{\Gamma, \varphi(t, \mathbf{y})}.$$

where Γ is a Π_2-sequent, φ a Σ_1-formula and t is free for x in φ.

Theorem 1. *$\mathrm{I}\Sigma_1$ is Π_2-conservative over $\mathrm{I}\Sigma_1^R$.*

Proof. So, our aim is to prove that if $\mathrm{I}\Sigma_1 \vdash \pi$ then $\mathrm{I}\Sigma_1^R \vdash \pi$ whenever π is a Π_2-sentence. If $\mathrm{I}\Sigma_1 \vdash \pi$, then by induction on the length of such a proof we see that some sequent Σ, π is provable in the pure predicate calculus. Here Σ is a finite set of negations of axioms of $\mathrm{I}\Sigma_1$. By the Cut-elimination theorem for the Tait calculus we know that there exists a cut-free derivation of the sequent. Thus we also have the sub-formula property (modulo substitution of terms) for our cut-free proof of Σ, π.

The proof is concluded by showing by induction on the length of cut-free derivations that if a sequent of the form Σ, Π is derivable then $\mathrm{I}\Sigma_1^R \vdash \Pi$. Here Σ is a finite set of negations of induction axioms of Σ_1-formulas and Π is a finite non-empty set of Π_2-formulas.

The basis case is trivial. So, for the inductive step, suppose we have a cut-free proof of Σ, Π. What can be the last step in the proof of this sequent? Either the last rule yielded something in the Π-part of the sequent or in the Σ-part of it. In the first case nothing interesting happens and we almost automatically obtain the desired result by the induction hypothesis.

So, suppose something had happened in the Σ-part. All formulas in this part are of the form $\exists\, a\ [\varphi(0) \wedge \forall\, x\ (\varphi(x) \rightarrow \varphi(x+1)) \wedge \neg\varphi(a)]$, with $\varphi \in \Sigma_1$.

The last deduction step thus must have been the introduction of the existential quantifier and we can by a one step shorter proof derive for some term t the following sequent.

$$\Sigma', \varphi(0) \wedge \forall\, x\ (\varphi(x) \rightarrow \varphi(x+1)) \wedge \neg\varphi(t), \Pi$$

[1] The same fragment as in Definition 2.

By the inversion property of the Tait calculus (for a proof and precise formulation of the statement consult e.g. [Sch77] page 873) we obtain proofs of the same length of the following sequents

$$\Sigma', \varphi(0), \Pi \ , \quad \Sigma', \forall x\ (\varphi(x) \to \varphi(x+1)), \Pi \quad \text{and} \quad \Sigma', \neg\varphi(t), \Pi.$$

As all of $\varphi(0), \forall x\ (\varphi(x) \to \varphi(x+1))$ and $\neg\varphi(t)$ are Π_2-formulas, we can apply the induction hypothesis to conclude that we have the following.

$$\begin{array}{ll} I\Sigma_1^R \vdash \varphi(0), \Pi & (1) \\ I\Sigma_1^R \vdash \forall x\ (\varphi(x) \to \varphi(x+1)), \Pi & (2) \\ I\Sigma_1^R \vdash \neg\varphi(t), \Pi & (3) \end{array}$$

Recall that Π consists of Π_2-statements. So, we can apply the Σ_1-induction rule to (1) and (2) and obtain $\varphi(t), \Pi$. This together with (3) yields by one application of the cut rule (in $I\Sigma_1^R$) the desired result, that is, $I\Sigma_1^R \vdash \Pi$.

Corollary 1. $I\Sigma_n$ *is* Π_{n+1}*-conservative over* $I\Sigma_n^R$.

Proof. $I\Sigma_n^R$ is defined as the canonical generalization of Definition 3. Changing the indices in the proof of Theorem 1 immediately yields the result.

In [Bek99] this result is stated as Corollary 4.8. It is a corollary of his Reduction property, Theorem 2, which is also formalizable in the presence of the superexponential function. The proof of Parsons' theorem we have presented here is very close to the proof of the reduction property.

4. A model theoretic proof of Parsons' theorem

In this section we shall give a model theoretic proof of Parsons' theorem. Our proof has the following outline.

In Subsection 4.1 we give a slightly renewed proof of a theorem by Gaifman and Dimitracopoulos. This theorem says that under certain conditions a definitional extension of a theory has nice properties, like proving enough induction.

In Subsection 4.2 we use this theorem to give a characterization of $I\Sigma_1$ in terms of PRA and closure under iteration of a certain class of functions. In Theorem 4 we will see what it takes for a model $\mathcal{M}$ of PRA to also be a model of $I\Sigma_1$: A class of functions of this model should be majorizable by another class of functions.

This theorem is at the heart of our model theoretic proof of Parsons' theorem in Subsection 4.3. We will show that any countable model $\mathcal{N}$ of PRA falsifying $\pi \in \Pi_2$ can be extended to a countable model $\mathcal{N}'$ of $I\Sigma_1 + \neg\pi$ whence $I\Sigma_1 \nvdash \pi$. In extending the model we will, having Theorem 4 in the back of our mind, repeatedly majorize functions to finally obtain a model of $I\Sigma_1 + \neg\pi$.

Our proof is based on a proofsketch in an unpublished note of Visser ([Vis90]). The very same note inspired Zambella in his [Zam96] for a proof of a conservation result of Buss' S^1_2 over PV.

First, we fix some formulation of PRA and $\mathrm{I}\Sigma_1$ that suits the purposes of this section.

Definition 4. *The* language *of* PRA *is the language of* PA *plus a family of new function symbols* $\{\mathsf{Sup}_n \mid n{\in}\omega\}$. *The* non-logical axioms of PRA *come in three sorts.*

- *Defining axioms for* $+$, $\cdot$, *and* $<$,[2]
- *Defining axioms for the new symbols*
  - $\forall x\ \mathsf{Sup}_0(x) = 2x$,
  - $\{\mathsf{Sup}_{n+1}(0) = 1\}$,
  - $\{\forall x\ \mathsf{Sup}_{n+1}(x+1) = \mathsf{Sup}_n(\mathsf{Sup}_{n+1}(x)) \mid n \in \omega\}$,
- *Induction axioms for* $\Delta_0(\{\mathsf{Sup}_i\}_{i\in\omega})$*-formulas in the following form:* $\forall x\ (\varphi(0) \wedge \forall y{<}x\ (\varphi(y) \rightarrow \varphi(y+1)) \rightarrow \varphi(x))$.

The logical axioms *and* rules *are just as usual.*

The functions Sup_i describe on the standard model a well-known hierarchy; Sup_0 is the doubling function, Sup_1 is the exponentiation function, Sup_2 is superexponentiation, Sup_3 is superduperexponentiation and so on. It is also known that the Sup_i form an envelope for PRA, that is, every provably total recursive function of PRA gets eventually majorized by some Sup_i. (Essentially this is Parikh's theorem [Par71].) Consequently all terms of PRA are majorizable by a strictly monotone one.

PRA proves all the evident properties of the Sup_i functions like $\mathsf{Sup}_n(1) = 2$, $1 \leq \mathsf{Sup}_{n+1}(y)$, $x \leq y \rightarrow \mathsf{Sup}_n(x) \leq \mathsf{Sup}_n(y)$, $n{\leq}m \rightarrow \mathsf{Sup}_n(x){\leq}\mathsf{Sup}_m(y)$ and so on. Of course PRA proves in a trivial way the totality of all the Sup_i as these symbols form part of our language. We have chosen an equivalent variant of the usual induction axiom so that we end up with a Π_1-axiomatization of PRA. It is easy to see that our definition of PRA is equivalent, or more precisely equi-interpretable, to any other of our definitions of PRA.

Definition 5. *The theory* $\mathrm{I}\Sigma_1$ *is the theory that is obtained by adding to* PRA *induction axioms* $\varphi(0) \wedge \forall x\ (\varphi(x) \rightarrow \varphi(x+1)) \rightarrow \forall x\ \varphi(x)$ *for all* $\Sigma_1(\{\mathsf{Sup}_i\}_{i\in\omega})$*-formulas* $\varphi(x)$ *that may contain additional parameters.*

Reading conventions Throughout this section we will adhere to the following notational convention. Arithmetical formulas defining the graph of a function are denoted by lowercase greek letters. The corresponding lower case roman letter is reserved to be the symbol that refers to the function described by its graph. By the corresponding upper case roman letter we will denote the very short formula that defines the graph using the lower

[2] We can take for example Kaye's system PA^- from [Kay91] where in Ax 13 we replace the unbounded existential quantifier by a bounded one.

case roman letter and the identity symbol only. Context, like indices and so forth, are inherited in the expected way.

For example, if $\chi_n(x, y)$ is an arithmetical formula describing a function, in a richer language this function will be referred to by the symbol g_n. The corresponding G_n will refer to the simple formula $g_n(x) = y$ in the enriched language.

4.1. Introducing a new function symbol

In our discussion we shall like to work with a theory that arises as an extension of PRA by a definition. We will add a new function symbol f to the language of PRA together with the axiom φ that defines f. Moreover we would like to employ induction that involves this new function symbol, possibly also in the binding terms of the bounded quantifiers. We will see that if the function f allows for a simple definition and has some nice properties we have indeed access to the extended form of induction.

Essentially the justification boils down to a theorem of Gaifman and Dimitracopoulos [GD82] a proof of which can also be found in [HP93] (Theorem 1.48 and Proposition 1.3). We will closely follow here a proof of Beklemishev from [Bek97] which we slightly improved and modified.

We first give the necessary definitions before we come to formulate the main result, Theorem 2

Definition 6 ($\Delta_0(\{g_i\}_{i\in I})$-formulas, $\mathrm{I}\Delta_0(\{g_i\}_{i\in I})$).
Let $\{g_i\}_{i\in I}$ be a set of function symbols. The $\Delta_0(\{g_i\}_{i\in I})$-formulas are the bounded formulas in the language of PA *enriched with the function symbols $\{g_i\}_{i\in I}$. The new function symbols are also allowed to occur in the binding terms of the bounded quantifiers. By $\mathrm{I}\Delta_0(\{g_i\}_{i\in I})$ we mean the theory that comprises*

- *some open axioms describing some minimal arithmetic*[3]*,*
- *induction axioms for all $\Delta_0(\{g_i\}_{i\in I})$-formulas and*
- *(possibly) defining axioms of the symbols $\{g_i\}_{i\in I}$.*

The defining axioms of the symbols $\{g_i\}_{i\in I}$ are denoted by $\mathcal{D}(\{g_i\}_{i\in I})$.

From now on, we may thus write $\mathrm{I}\Delta_0(\{\mathsf{Sup}_i\}_{i\in\omega})$ instead of PRA.

Definition 7 ($\mathsf{Tot}(\varphi)$, $\mathsf{Mon}(\varphi)$).
Let $\varphi(x, y)$ be a $\Delta_0(\{g_i\}_{i\in I})$ formula. By $\mathsf{Tot}(\varphi)$ we shall denote the formula $\forall x\, \exists!\, y\ \varphi(x, y)$[4] *stating that φ can be regarded as a total function. By $\mathsf{Mon}(\varphi)$ we shall denote the formula $\forall x, x', y, y'\ (x \le x' \wedge \varphi(x, y) \wedge \varphi(x', y') \to y \le y') \wedge \mathsf{Tot}(\varphi)$ stating the monotonicity of the total φ.*

[3] For example the open part of Robinson's arithmetic.

[4] That is, $\forall x\, \exists y\ \varphi(x, y) \wedge \forall x\, \forall y\, \forall y'\ (\varphi(x, y) \wedge \varphi(x, y') \to y = y')$.

Definition 8 ($\Delta_0(\{g_i\}_{i\in I}, F)$-formula, $\mathrm{I}\Delta_0(\{g_i\}_{i\in I}, F)$).
Let φ be such that $\mathrm{I}\Delta_0(\{g_i\}_{i\in I}) \vdash \mathsf{Tot}(\varphi)$. Recall that the uppercase letter F paraphrases the formula $f(x) = y$. A $\Delta_0(\{g_i\}_{i\in I}, F)$-formula is a $\Delta_0(\{g_i\}_{i\in I})$-formula possibly containing occurrences of F. By $\mathrm{I}\Delta_0(\{g_i\}_{i\in I}, F)$ we denote the theory $\mathrm{I}\Delta_0(\{g_i\}_{i\in I})$ where we now also have induction for $\Delta_0(\{g_i\}_{i\in I}, F)$ formulas. The defining axiom of f, in our case φ, is also in $\mathrm{I}\Delta_0(\{g_i\}_{i\in I}, F)$.

Note that f cannot occur in a bounding term in an induction axiom of $\mathrm{I}\Delta_0(\{g_i\}_{i\in I}, F)$. Also note that F is nothing but a formula containing f stating $f(x) = y$ and consists of just six symbols (if f is unary). Of course later we will substitute for F an arithmetical definition of the graph of f, that is, $\varphi(x, y)$.

The main interest of the extension of $\mathrm{I}\Delta_0(\{g_i\}_{i\in I})$ by a definition of f is in Theorem 2 and in its Corollary 2. The latter says that we can freely use $f(x)$ as an abbreviation of $\varphi(x,y)$ and have access to $\Delta_0(\{g_i\}_{i\in I}, f)$-induction whenever f has a $\Delta_0(\{g_i\}_{i\in I})$ graph and is provably total and monotone in $\mathrm{I}\Delta_0(\{g_i\}_{i\in I})$.

First we prove some technical but rather useful lemmata. They are slight improvements of Beklemishev's Lemma 5.12 and 5.13 from [Bek97]. From now on we will work under the assumptions of Theorem 2 so that $\mathrm{I}\Delta_0(\{g_i\}_{i\in I})$ is such that any term t in its language is provably majorizable by some other term $\tilde{t}$ that is strictly increasing in all of its arguments. Throughout the forthcoming proofs we will for any term t denote by $\tilde{t}$ such a term that is provably strictly monotone (in all of its arguments) and majorizing t.

Lemma 2. *For every term $s(\mathbf{a})$ of $\mathrm{I}\Delta_0(\{g_i\}_{i\in I}, f)$ and every $R \in \{\leq, \geq, =, <, >\}$ there are terms t^R_s and $\tilde{s}(a)$ strictly increasing in all of their arguments and a $\Delta_0(\{g_i\}_{i\in I}, F)$-formula $\psi^R_s(\mathbf{a}, b, y)$ such that $\mathrm{I}\Delta_0(\{g_i\}_{i\in I}, F){+}\mathsf{Mon}(\varphi) \vdash \forall\, y{\geq}t^R_s(\mathbf{a})\ (s(\mathbf{a})Rb \leftrightarrow \psi^R_s(\mathbf{a}, b, y))$ and $\mathrm{I}\Delta_0(\{g_i\}_{i\in I}, F){+}\mathsf{Mon}(\varphi) \vdash \forall\, \mathbf{x}\ (s(\mathbf{x}) \leq \tilde{s}(\mathbf{x}))$.*

Proof. The proof proceeds by induction on $s(\mathbf{a})$. In the basis case nothing has to be done as x_iRb, $0Rb$ and $1Rb$ are all atomic $\Delta_0(\{g_i\}_{i\in I}, F)$-formulas. Moreover all of the x_i, 0 and 1 are (provably) strictly monotone in all of their arguments. For the induction case consider $s(\mathbf{a}) = h(s_1(\mathbf{a}))$, where h is either one of the g_i or $h = f$. For simplicity we assume here that h is a unary function.

The induction hypothesis provides us with a $\Delta_0(\{g_i\}_{i\in I}, F)$-formula $\psi^=_{s_1}(\mathbf{a}, b, y)$ and terms $t^=_{s_1}(\mathbf{a})$ and $\tilde{s}_1(\mathbf{a})$ such that

$$\mathrm{I}\Delta_0(\{g_i\}_{i\in I}, F) + \mathsf{Mon}(\varphi) \vdash \forall\, y{\geq}t^=_{s_1}(\mathbf{a})\ (s_1(\mathbf{a}) = b \leftrightarrow \psi^=_{s_1}(\mathbf{a}, b, y)),$$

and

$$\mathrm{I}\Delta_0(\{g_i\}_{i\in I}, F) + \mathsf{Mon}(\varphi) \vdash \forall\, \mathbf{x}\ (s_1(\mathbf{x}) \leq \tilde{s}_1(\mathbf{x})).$$

We now want to say that $h(s_1(\mathbf{a}))Rb$ in a $\Delta_0(\{g_i\}_{i\in I}, F)$ way. This can be done by $\exists\, y', y''\leq y\ (\psi^{=}_{s_1}(\mathbf{a}, y', y) \wedge h(y') = y'' \wedge y''Rb)$ whenever $y \geq t^{=}_{s_1}(\mathbf{a}) + \tilde{s}(\mathbf{a})$. Here we define $\tilde{s}(\mathbf{a})$ to be just $f(\tilde{s}_1(\mathbf{a}))$ in case $h = f$ and $\tilde{g}_i(\tilde{s}_1(\mathbf{a}))$ in case $h = g_i$. Clearly $\mathrm{I}\Delta_0(\{g_i\}_{i\in I}, F) + \mathsf{Mon}(\varphi) \vdash \forall\, \mathbf{x}\ (s(\mathbf{x}) \leq \tilde{s}(\mathbf{x}))$. Indeed one easily sees that

$$\mathrm{I}\Delta_0(\{g_i\}_{i\in I}, F) + \mathsf{Mon}(\varphi) \vdash \forall\, y{\geq}t^{=}_{s_1}(\mathbf{a}) + \tilde{s}(\mathbf{a})\ \ [h(s_1(\mathbf{a}))Rb \leftrightarrow \exists\, y', y''{\leq}y\ (\psi^{=}_{s_1}(\mathbf{a}, y', y) \wedge h(y') = y'' \wedge y''Rb)].$$

It is also easy to see that $t^{=}_{s_1}(\mathbf{a}) + \tilde{s}(\mathbf{a})$ is indeed monotone. In case $h = f$ we need $\mathsf{Mon}(\varphi)$ here.

A similar reduction applies to the case when the function g has more than one argument.

It is possible to simplify the above reduction a bit by distinguishing between $h = f$ and $h \neq f$ and also $R ==$ and $R \neq=$, or by proving the lemma just for $R ==$ and showing that all the other cases can be reduced to this. We are not very much interested in optimality at this point though.

Lemma 3. *For every $\Delta_0(\{g_i\}_{i\in I}, f)$-formula $\theta(\mathbf{a})$ there is a $\Delta_0(\{g_i\}_{i\in I}, F)$-formula $\theta_0(\mathbf{a}, y)$ and a provably monotonic term $t_\theta(\mathbf{a})$ such that*
$\mathrm{I}\Delta_0(\{g_i\}_{i\in I}, F) + \mathsf{Mon}(\varphi) \vdash \forall\, y{\geq}t_\theta(\mathbf{a})\ \ (\theta(\mathbf{a}) \leftrightarrow \theta_0(\mathbf{a}, y))$.

Proof. The lemma is proved by induction on θ.

- Basis. In this case $\theta(\mathbf{a})$ is $s_1(\mathbf{a})R\, s_2(\mathbf{a})$. Applying Lemma 2 we see that[5] $s_1(\mathbf{a})R\, s_2(\mathbf{a}) \leftrightarrow \exists\, b{\leq}y\ (\psi^{=}_{s_2}(\mathbf{a}, b, y) \wedge \psi^{R}_{s_1}(\mathbf{a}, b, y))$ whenever $y \geq t_{s_1}(\mathbf{a}) + t_{s_2}(\mathbf{a})$.
- The only interesting induction case is where a bounded quantifier is involved. We consider the case when $\theta(\mathbf{a})$ is $\exists\, x{\leq}s(\mathbf{a})\ \xi(\mathbf{a}, x)$. The induction hypothesis yields a provably monotonic term $t_\xi(\mathbf{a}, x)$ and a $\Delta_0(\{g_i\}_{i\in I}, F)$-formula $\xi_0(\mathbf{a}, x, y)$ such that provably

  $$\forall\, y{\geq}t_\xi(\mathbf{a}, x)\ (\xi(\mathbf{a}, x) \leftrightarrow \xi_0(\mathbf{a}, x, y))$$

  . Combining this with Lemma 2 we get that provably

  $$\exists\, x{\leq}s(\mathbf{a})\ \xi(\mathbf{a}, x) \leftrightarrow \exists\, x'{\leq}y\ (\psi^{=}_{s}(\mathbf{a}, x', y) \wedge \exists\, x{\leq}x'\ \xi_0(\mathbf{a}, x, y))^{6}$$

  whenever $y \geq \tilde{s}(\mathbf{a}) + t^{=}_{s}(\mathbf{a}) + t_\xi(\mathbf{a}, \tilde{s}(\mathbf{a}))$.

Theorem 2. *Let $\mathrm{I}\Delta_0(\{g_i\}_{i\in I})$ be such that any term t in its language is provably majorizable by some other term $\tilde{t}$ that is strictly increasing in all of its arguments. We have that $\mathrm{I}\Delta_0(\{g_i\}_{i\in I}, F) + \mathsf{Mon}(\varphi) \vdash \mathrm{I}\Delta_0(\{g_i\}_{i\in I}, f)$.*

[5] If we only want to use Lemma 2 with R being $=$ we can observe that $s_1(\mathbf{a})R\, s_2(\mathbf{a}) \leftrightarrow \exists\, b, c{\leq}y\ (\psi^{=}_{s_1}(\mathbf{a}, b, y) \wedge \psi^{=}_{s_2}(\mathbf{a}, c, y) \wedge bRc)$ whenever $y \geq t_{s_1}(\mathbf{a}) + t_{s_2}(\mathbf{a})$.

[6] Alternatively, one could take $\exists\, x{\leq}y\ (\psi^{\geq}_{s}(\mathbf{a}, x, y) \wedge \xi_0(\mathbf{a}, x, y))$ for $y \geq t^{\geq}_{s}(\mathbf{a}) + t_\xi(\mathbf{a}, \tilde{s}(\mathbf{a}))$.

Proof. We will prove the least number principle for $\Delta_0(\{g_i\}_{i\in I}, f)$-formulas in $\mathrm{I}\Delta_0(\{g_i\}_{i\in I}, F) + \mathsf{Mon}(\varphi)$ as this is equivalent to induction for $\Delta_0(\{g_i\}_{i\in I}, f)$-formulas. So, let $\theta(x, \mathbf{a})$ be a $\Delta_0(\{g_i\}_{i\in I}, f)$-formula and reason in $\mathrm{I}\Delta_0(\{g_i\}_{i\in I}, F) + \mathsf{Mon}(\varphi)$. By Lemma 3 we have a strict monotonic term $t_\theta(x, \mathbf{a})$ and a $\Delta_0(\{g_i\}_{i\in I}, F)$-formula $\theta_0(x, \mathbf{a}, y)$ such that $\theta(x, \mathbf{a}) \leftrightarrow \theta_0(x, \mathbf{a}, y)$ whenever $y \geq t_\theta(x, \mathbf{a})$.

Now assume $\exists\, x\ \theta(x, \mathbf{a})$. We will show that $\exists\, x\ (\theta(x, \mathbf{a}) \wedge \forall\, x'{<}x\ \neg\theta(x', \mathbf{a}))$. Let x be such that $\theta(x, \mathbf{a})$. We now fix some $y \geq t_\theta(x, \mathbf{a})$. Thus we have $\theta_0(x, \mathbf{a}, y)$. Applying the least number principle to the $\Delta_0(\{g_i\}_{i\in I}, F)$-formula $\theta_0(x, \mathbf{a}, y)$ we get a minimal x_0 such that $\theta_0(x_0, \mathbf{a}, y)$. As $x_0 < x$ and t_θ is monotone we have $y \geq t_\theta(x, \mathbf{a}) \geq t_\theta(x_0, \mathbf{a})$ and thus $\theta(x_0, \mathbf{a})$. If now $x' < x_0$ such that $\theta(x', \mathbf{a})$ then also $\theta_0(x', \mathbf{a}, y)$ which would conflict the minimality of x_0 for θ_0. Thus x_0 is the minimal element such that $\theta(x_0, \mathbf{a})$.

As in [Bek97] (Remark 5.14) we note here that Theorem 2 shows that $\Delta_0(\{g_i\}_{i\in I}, f)$-induction is actually provable from $\Delta_0(\{g_i\}_{i\in I}, F)$-induction where the bounding terms are just plain variables. Also we note that Lemma 2 and Lemma 3 do not use the full strength of $\mathrm{I}\Delta_0(\{g_i\}_{i\in I}, F)$.

Corollary 2. *Let $\mathrm{I}\Delta_0(\{g_i\}_{i\in I})$ be such that any term t in its language is provably majorizable by some other term $\tilde{t}$ that is strictly increasing in all of its arguments. Let f be $\Delta_0(\{g_i\}_{i\in I})$-definable by φ. Then, $\mathrm{I}\Delta_0(\{g_i\}_{i\in I}) + \mathsf{Mon}(\varphi) \vdash \mathrm{I}\Delta_0(\{g_i\}_{i\in I}, f)$.*

Proof. Immediate from Theorem 2 by replacing every occurrence of F by φ.

4.2. PRA*,* $\mathrm{I}\Sigma_1$ *and iterations of total functions*

This subsection contains two main results. In Theorem 3 we shall characterize the difference between $\mathrm{I}\Sigma_1$ and PRA in terms of provable closure of iteration of a certain class of functions.

In Theorem 4 we use this characterization to give a sufficient condition for a model of PRA to be also a model of $\mathrm{I}\Sigma_1$.

Let us first specify what we mean by function iteration. If f denotes a function we will denote by f^{it} the (unique) function satisfying the following primitive recursive schema: $f^{\mathsf{it}}(0){=}1$, $f^{\mathsf{it}}(x+1){=}f(f^{\mathsf{it}}(x))$.

Definition 9. *Let $\varphi(x, y)$ be some formula. By $\varphi^{\mathsf{it}}(x, y)$ we denote $\exists\, \sigma\ \tilde{\varphi}^{\mathsf{it}}(\sigma, x, y)$ where $\tilde{\varphi}^{\mathsf{it}}(\sigma, x, y)$ is the formula $\mathsf{Finseq}(\sigma) \wedge \mathsf{lh}(\sigma) = x + 1 \wedge \sigma_0 = 1 \wedge \sigma_x = y \wedge \forall\, i{<}x\ \varphi(\sigma_i, \sigma_{i+1})$.*

Note that if PRA proves the functionality of a $\Delta_0(\{\mathsf{Sup}_i\}_{i\in\omega})$-formula φ, it also proves the functionality of $\tilde{\varphi}^{\mathsf{it}}$, for example by proving by induction on σ that $\forall\, \sigma\, \forall\, x, y, y', \sigma'{\leq}\sigma\ (\tilde{\varphi}^{\mathsf{it}}(\sigma, x, y) \wedge \tilde{\varphi}^{\mathsf{it}}(\sigma', x, y') \rightarrow \sigma = \sigma' \wedge y = y')$.

As we will need upperbounds on sequences of numbers a short remark on coding is due here. By $[a_0, \ldots, a_n]$ we will denote the code of the sequence $a_0, \ldots, a_n$ of natural numbers via some fixed coding technique. By

$[a_0,\ldots,a_n]\sqcap[b_0,\ldots,b_m]$ we will denote the code of the sequence $a_0,\ldots,a_n,b_0,\ldots,b_m$ that arises from concatenating $b_0,\ldots,b_m$ to $a_0,\ldots,a_n$ (to the right).

The projection functions are referred to by sub indexing. So, σ_i will be a_i if $\sigma=[a_0,\ldots,a_n]$ and $i\leq n$ and zero if $i>n$, and $n+1$ is called the length of σ. We say that σ is an initial subsequence of σ' if $\sigma=[a_0,\ldots a_n]$ and $\sigma'=[a_0,\ldots a_n,\ldots a_m]$ and $m\geq n$. We denote this by $\sigma\sqsubseteq\sigma'$.

Further, we shall employ well known expressions like $\mathsf{lh}(\sigma)$, giving the length of a sequence σ. If we write down statements involving sequences we will tacitly assume that the statements actually make sense. For example, $\forall\, i{<}\mathsf{lh}(x)\ \psi$ will thus actually denote $\mathsf{Finseq}(x)\wedge\forall\, i{<}\mathsf{lh}(x)\ \psi$.

We shall not fix any specific coding protocol as any protocol with elementary projections, concatenation etcetera is good for us.

The following theorem tells us what is the difference between PRA and IΣ_1 in terms of totality statements of $\Delta_0(\{\mathsf{Sup}_i\}_{i\in\omega})$-definable functions.

Theorem 3. $\mathrm{I}\Sigma_1\equiv\mathrm{PRA}+\{\mathsf{Tot}(\varphi)\rightarrow\mathsf{Tot}(\varphi^{\mathsf{it}})\mid\varphi\in\Delta_0(\{\mathsf{Sup}_i\}_{i\in\omega})\}$.

Proof. For one inclusion we only need to show that $\mathrm{I}\Sigma_1\vdash\mathsf{Tot}(\varphi)\rightarrow\mathsf{Tot}(\varphi^{\mathsf{it}})$ but this follows easily from a Σ_1-induction on x in $\exists\,\sigma\,\exists\,y\ \tilde{\varphi}^{\mathsf{it}}(\sigma,x,y)$ under the assumption that $\forall\,x\,\exists\,y\ \varphi(x,y)$. We shall thus concentrate on the harder direction $\mathrm{PRA}+\{\mathsf{Tot}(\varphi)\rightarrow\mathsf{Tot}(\varphi^{\mathsf{it}})\mid\varphi\in\Delta_0(\{\mathsf{Sup}_i\}_{i\in\omega})\}\vdash\mathrm{I}\Sigma_1$.

To this end we reason in $\mathrm{PRA}+\{\mathsf{Tot}(\varphi)\rightarrow\mathsf{Tot}(\varphi^{\mathsf{it}})\mid\varphi\in\Delta_0(\{\mathsf{Sup}_i\}_{i\in\omega})\}$ and assume $\exists\,y\ \ \psi(0,y)\wedge\forall\,x\ \ (\exists y\ \ \psi(x,y)\ \rightarrow\ \exists\,y\ \ \psi(x+1,y))$ for some $\Delta_0(\{\mathsf{Sup}_i\}_{i\in\omega})$-formula $\psi(x,y)$. Our aim is to obtain $\forall\,x\,\exists\,y\ \psi(x,y)$.

Let $\mathsf{Least}_{\psi,x}(y)$ denote the formula $\psi(x,y)\wedge\forall\,y'{<}y\ \neg\psi(x,y')$. We are going to define in a $\Delta_0(\{\mathsf{Sup}_i\}_{i\in\omega})$-way a formula $\varphi(x,y)$ so that $f^{\mathsf{it}}(x{+}1)=[y_0,\cdots,y_x]$ with $\forall\,i{\leq}x\ \mathsf{Least}_{\psi,i}(y_i)$.

$$\varphi(x,y):=\begin{cases}(i)\ \ \ (x=0\wedge y=0) & \vee\\ (ii)\ \ (x=1\wedge\exists\,y'{<}y\ (y=[y']\wedge\mathsf{Least}_{\psi,0}(y'))) & \vee\\ (iii)\ (x>1\wedge\forall\,i{<}\mathsf{lh}(x)\ \mathsf{Least}_{\psi,i}(x_i)\wedge & \\ \qquad\exists\,y'{<}y\ (y=x\sqcap[y']\wedge\mathsf{Least}_{\psi,\mathsf{lh}(x)}(y'))) & \vee\\ (iv)\ (x>1\wedge\neg(\forall\,i{<}\mathsf{lh}(x)\ \mathsf{Least}_{\psi,i}(x_i))\wedge y=0) & \end{cases}$$

Thus, the function f defined by φ has the following properties. It is always zero unless $x{=}1$ or x is of the form $[y_0,\cdots,y_n]$ where each y_i is the smallest witness for $\exists y\ \psi(i,y)$.

We note, that by our assumptions $\exists y\,\psi(0,y)$ and $\forall x\,(\exists y\,\psi(x,y)\rightarrow\exists y\,\psi(x{+}1,y))$, the function f is total. As the definition of φ is clearly $\Delta_0(\{\mathsf{Sup}_i\}_{i\in\omega})$ we may conclude $\mathsf{Tot}(f^{\mathsf{it}})$.

We shall show that f^{it} is $\Delta_0(\{\mathsf{Sup}_i\}_{i\in\omega})$-definable, and that provably $\mathsf{Mon}(f^{\mathsf{it}})$. If we know this, then our result follows immediately. Because, by an easy $\mathrm{I}\Delta_0(\{\mathsf{Sup}_i\}_{i\in\omega},f^{\mathsf{it}})$-induction we conclude $\forall x\ \psi(x,(f^{\mathsf{it}}(x{+}1))_x)$, whence $\forall x\,\exists y\ \psi(x,y)$. By Corollary 2 we conclude $\mathrm{PRA}+\{\mathsf{Tot}(\varphi)\rightarrow\mathsf{Tot}(\varphi^{\mathsf{it}})\mid\varphi\in\Delta_0(\{\mathsf{Sup}_i\}_{i\in\omega})\}\vdash\forall x\,\exists y\ \psi(x,y)$ and we are done.

We will first see inside our theory that $\mathsf{Mon}(f^{\mathsf{it}})$. The monotonicity of f^{it} is intuitively clear but we have to show that we can catch this intuition using only $\Delta_0(\{\mathsf{Sup}_i\}_{i\in\omega})$-induction.

For example, we can first prove by induction on x that all of the $f^{\mathsf{it}}(x{+}1)$ are "good sequences" where by a good sequence we mean one of the form $[y_0,\ldots,y_x]$ with the y_i minimal witnesses to $\exists y\ \psi(i,y)$. To make this a $\Delta_0(\{\mathsf{Sup}_i\}_{i\in\omega})$-induction we should reformulate the statement as for example $\forall z\,\forall\,\sigma,x,y{\leq}z\ (\tilde{\varphi}^{\mathsf{it}}(\sigma,x+1,y)\rightarrow \mathsf{Goodseq(y)})$.

Now assume $\tilde{\varphi}^{\mathsf{it}}(\sigma',x',y')$. We will show by induction on x that

$$\forall\,x{\leq}x'\ \exists\,\sigma{\leq}\sigma'\ \exists\,y{\leq}y'\ \tilde{\varphi}^{\mathsf{it}}(\sigma,x'-x,y) \qquad (+)$$

from which monotonicity follows. If $x=0$ we take $\sigma'=\sigma$ and $y=y'$. For the inductive step, let $\sigma\leq\sigma'$ and $y\leq y'$ be such that $\tilde{\varphi}^{\mathsf{it}}(\sigma,x'-x,y)$. We assume that $x+1\leq x'$ hence $\mathsf{lh}(\sigma)>1$, for if not, the solution is trivial.

By σ_{-1} we denote the sequence that is obtained from σ by deleting the last element. Clearly $\tilde{\varphi}^{\mathsf{it}}(\sigma_{-1},x'{-}x{-}1,(\sigma_{-1})_{x'-x-1})$ and $\varphi((\sigma_{-1})_{x'-x-1},y)$. Thus $(\sigma_{-1})_{x'-x-1}$ is a good sequence which implies that clause (iii) in the definition of φ is used to determine y. Consequently $(\sigma_{-1})_{x'-x-1}\sqsubseteq y$ and thus $(\sigma_{-1})_{x'-x-1}\leq y\leq y'$. Moreover we note that $\sigma_{-1}\sqsubseteq\sigma$ and thus $\sigma_{-1}\leq\sigma\leq\sigma'$.

We now want to show the $\Delta_0(\{\mathsf{Sup}_i\}_{i\in\omega})$-ness of $\varphi^{\mathsf{it}}(x,y)$ by providing an upperbound on the σ in $\tilde{\varphi}^{\mathsf{it}}(\sigma,x,y)$. Under any reasonable choice of our coding machinery, we can find an $n\in\omega$ such that

$$\begin{array}{ll}(a) & [\overbrace{y,\cdots,y}^{x\text{ times}}]\leq\mathsf{Sup}_n(x+y),\\ (b) & \mathsf{Sup}_n(x+y)\sqcap[y]\leq\mathsf{Sup}_n(x+y+1).\end{array}$$

For such an n it is not hard to see that

$$\exists\sigma\ \tilde{\varphi}^{\mathsf{it}}(\sigma,x,y)\leftrightarrow\exists\,\sigma'{\leq}\mathsf{Sup}_n(x+y)\ \tilde{\varphi}^{\mathsf{it}}(\sigma,x,y).$$

This, we see by proving by induction on σ that

$$\forall\sigma\,\forall\,x,y{\leq}\sigma\ (\tilde{\varphi}^{\mathsf{it}}(\sigma,x,y)\rightarrow\exists\,\sigma'{\leq}\mathsf{Sup}_n(x+y)\ \tilde{\varphi}^{\mathsf{it}}(\sigma,x,y)).$$

We note that this is sufficient as $\tilde{\varphi}^{\mathsf{it}}(\sigma,x,y)\rightarrow x,y\leq\sigma$. The only interesting possibility in the induction step is when we get for some new $x+1,y$ that $\tilde{\varphi}^{\mathsf{it}}(\sigma+1,x+1,y)$. For $\sigma'':=(\sigma+1)_{-1}$ we have that $\sigma''<\sigma+1$ and $\tilde{\varphi}^{\mathsf{it}}(\sigma'',x,y_{-1})$. By the induction hypothesis we may assume that $\sigma''\leq\mathsf{Sup}_n(x+y_{-1})$. By the definition of $\tilde{\varphi}^{\mathsf{it}}$, we now see that $\tilde{\varphi}^{\mathsf{it}}(\sigma''\sqcap[y],x{+}1,y)$. But,

$$\begin{aligned}\sigma''\sqcap[y]&\leq\mathsf{Sup}_n(x+y_{-1})\sqcap[y]\\&\leq\mathsf{Sup}_n(x+y)\sqcap[y]\\&\leq\mathsf{Sup}_n(x+y+1).\end{aligned}$$

We note that we filled the gap between PRA and IΣ_1 by transforming an admissible rule of PRA to axiom form. Indeed $\mathsf{Tot}(\varphi) \mathrel{|\!\!\sim} \mathsf{Tot}(\varphi^{\mathsf{it}})$ is an admissible rule of PRA. For if PRA $\vdash \mathsf{Tot}(\varphi)$, then f is a primitive recursive function as is well known. But f^{it} is constructed from f by a simple recursion. Thus f^{it} is primitive recursive and hence provably total in PRA. The same phenomenon occurs in passing from IΣ_1^R to IΣ_1 where the (trivially) admissible Σ_1-induction rule is added in axiom form to PRA to obtain IΣ_1.

The fact that we allow for variables in Theorem 3 is essential. For if not, the logical complexity of PRA + $\{\mathsf{Tot}(\varphi) \to \mathsf{Tot}(\varphi^{\mathsf{it}}) \mid \varphi \in \Delta_0(\{\mathsf{Sup}_i\}_{i\in\omega})\}$ would be[7] Δ_3 and so would be the logical complexity of IΣ_1. But it is well known that IΣ_1 can not be proved by any consistent collection of Σ_3-sentences.

A parameter-free version of PRA+$\{\mathsf{Tot}(\varphi) \to \mathsf{Tot}(\varphi^{\mathsf{it}}) \mid \varphi \in \Delta_0^-(\{\mathsf{Sup}_i\}_{i\in\omega})\}$ will be equivalent to parameter-free Σ_1-induction, I$\Sigma_1{}^-$.

We now come to prove a theorem that tells us when a model of PRA is also a model of IΣ_1. This lemma is formulated in terms of majorizability behavior of some total functions. A total function of a model M is a relation $\varphi(x, y)$ (possibly with parameters from M) for which $M \models \mathsf{Tot}(\varphi)$. Often we will write $f \leq g$ as short for $\forall x\ (\exists y\ \varphi(x, y) \to \exists y'\ (\chi(x, y') \wedge y \leq y'))$ and say that f is majorized by g. Thus if $f \leq g$ we automatically have $\mathsf{Tot}(\varphi) \to \mathsf{Tot}(\chi)$.

Theorem 4. *Let $\mathcal{M}$ be a model of* PRA*. If every $\Delta_0(\{\mathsf{Sup}_i\}_{i\in\omega})$-definable total function (with parameters) of $\mathcal{M}$ is majorized by $m + \mathsf{Sup}_n$ for some $m \in \mathcal{M}$ and some $n \in \omega$, then $\mathcal{M}$ is also a model of* IΣ_1*.*

Proof. Let $\mathcal{M}$ be satisfying our conditions. To see that $\mathcal{M} \models$ IΣ_1 we need in the light of Theorem 3 to show that $\mathcal{M} \models \mathsf{Tot}(\varphi) \to \mathsf{Tot}(\varphi^{\mathsf{it}})$ for any $\Delta_0(\{\mathsf{Sup}_i\}_{i\in\omega})$ function φ with parameters in $\mathcal{M}$. So, we consider some function f such that $\mathcal{M} \models \mathsf{Tot}(\varphi)$. We choose $m \in \mathcal{M} \setminus \{0\}$ and $n \in \omega$ large enough so that

$(a.)$ $\mathcal{M} \models f \leq m + \mathsf{Sup}_n$,
$(b.)$ $\mathcal{M} \models \forall x\ (m + \mathsf{Sup}_{n+1}(mx + m + 1) \leq \mathsf{Sup}_{n+1}(mx + m + m))$.

The second condition is automatically satisfied if m is a non-standard element.

An easy $\Delta_0(\{\mathsf{Sup}_i\}_{i\in\omega})$-induction shows that $(m+\mathsf{Sup}_n)^{\mathsf{it}}(x) \leq \mathsf{Sup}_{n+1}(mx+m)$. (Remember that we have excluded $m = 0$.) The case $x = 0$ is trivial as

[7] Actually we should be more careful here as we work in a richer language. However this makes no essential difference as all the Sup_n are Δ_1-definable over EA.

$1 \leq \mathsf{Sup}_{n+1}(m)$. For the inductive step we see that[8]

$$\begin{array}{ll} (m+\mathsf{Sup}_n)^{\mathsf{it}}(x+1) & = \\ (m+\mathsf{Sup}_n)((m+\mathsf{Sup}_n)^{\mathsf{it}}(x)) & \leq_{\text{i.h.}} \\ m+\mathsf{Sup}_n(\mathsf{Sup}_{n+1}(mx+m)) & \leq_{\text{def.}} \\ m+\mathsf{Sup}_{n+1}(mx+m+1) & \leq_{(b.)} \\ \mathsf{Sup}_{n+1}(mx+m+m) = \mathsf{Sup}_{n+1}(m(x+1)+m). & \end{array}$$

We can use the obtained bounds to show the totality of f^{it} by estimating the size of σ that witnesses $\tilde{\varphi}^{\mathsf{it}}(\sigma, x, y)$. We know (outside PRA) that σ is of the form

$$\begin{array}{ll} [1, f(1), f(f(1)), \ldots, f^x(1)] & \leq \\ [1, m+\mathsf{Sup}_n(1), m+\mathsf{Sup}_n(f(1)), \ldots, m+\mathsf{Sup}_n(f^{x-1}(1))] & \leq \\ [1, m+\mathsf{Sup}_n(1), (m+\mathsf{Sup}_n)^2(1), \ldots, (m+\mathsf{Sup}_n)^2(f^{x-2}(1))] & \leq \\ \vdots & \vdots \\ [1, m+\mathsf{Sup}_n(1), (m+\mathsf{Sup}_n)^2(1), \ldots, (m+\mathsf{Sup}_n)^x(1)] & \leq \\ [(m+\mathsf{Sup}_n)^x(1), \ldots, (m+\mathsf{Sup}_n)^x(1)] & \leq \\ [\mathsf{Sup}_{n+1}(mx+m), \ldots, \mathsf{Sup}_{n+1}(mx+m)] & \end{array}$$

Every time we used dots here in our informal argument, some $\Delta_0(\{\mathsf{Sup}_i\}_{i\in\omega})$-induction should actually be applied. To neatly formalize our reasoning we choose some $k \in \omega$ large enough for our n and m such that (in $\mathcal{M}$)

(*c.*) $[1] \leq \mathsf{Sup}_{n+k}(2m)$
(*d.*) $\mathsf{Sup}_{n+k}(m(x+1)+m) \sqcap [\mathsf{Sup}_{n+1}(m(x+1)+m)] \leq \mathsf{Sup}_{n+k}(m(x+2)+m)$[9]

With these choices for m, n and k it is easy to prove by $\Delta_0(\{\mathsf{Sup}_i\}_{i\in\omega})$-induction that

$$\forall x\, \exists \sigma {\leq} \mathsf{Sup}_{n+k}(m(x+1)+m)\, \exists y {\leq} \mathsf{Sup}_{n+1}(mx+m)\ \tilde{\varphi}^{\mathsf{it}}(\sigma, x, y).$$

If $x = 0$ then $\tilde{\varphi}^{\mathsf{it}}([1], 0, 1)$ and by (*c.*) we have $[1] \leq \mathsf{Sup}_{n+k}(m(0+1)+m)$. Also $1 \leq \mathsf{Sup}_{n+1}(m)$. Now suppose $\tilde{\varphi}^{\mathsf{it}}(\sigma, x, y)$ with σ and y below their respective bounds. We have by the definition of $\tilde{\varphi}^{\mathsf{it}}$ that $\tilde{\varphi}^{\mathsf{it}}(\sigma \sqcap [f(y)], x+1, f(y))$ (again we do as if we had f available in our language). We need to show that the new values do not grow too fast. But,

$$\begin{array}{rll} f(y) & \leq_{\text{I.H.}} f(\mathsf{Sup}_{n+1}(mx+m)) & \leq_{(a.)} \\ m+\mathsf{Sup}_n(\mathsf{Sup}_{n+1}(mx+m)) & \leq_{(b.)} f(\mathsf{Sup}_{n+1}(m(x+1)+m)) & \end{array}$$

---

[8] This looks like a legitimate induction but remember that $(m+\mathsf{Sup}_n)^{\mathsf{it}}$ has an a priori $\Sigma_1(\{\mathsf{Sup}_i\}_{i\in\omega})$-definition. The argument should thus be encapsulated in a $\Delta_0(\{\mathsf{Sup}_i\}_{i\in\omega})$-induction, for example by proving $\forall z\, \forall \sigma, x, y {\leq} z\ (\widetilde{(m+\mathsf{Sup}_n)}^{\mathsf{it}}(\sigma, x, y) \to y \leq \mathsf{Sup}_{n+1}(mx+m))$. The essential reasoning though boils down to the argument given here.

[9] It is not hard to convince oneself that under any reasonable coding protocol such a k does exist.

as we have seen before. By (*d.*) we get that

$$\begin{aligned}\sigma \sqcap [f(y)] &\leq_{\text{I.H.}} \mathsf{Sup}_{n+k}(m(x+1)+m) \sqcap [\mathsf{Sup}_{n+1}(m(x+1)+m)] \\ &\leq_{(d.)} \mathsf{Sup}_{n+k}(m(x+2)+m).\end{aligned}$$

4.3. The actual proof of Parsons' theorem

In the setting of this section we formulate Parsons' theorem as follows.

Theorem 5. $\forall \pi \in \Pi_2$ $(\mathrm{I}\Sigma_1 \vdash \pi \Rightarrow \mathrm{PRA} \vdash \pi)$

Before we give the proof of Parsons' theorem we first agree on some model theoretic notation.

We recall the definition of M' being a 1-elementary extension of M, denoted by $M \prec_1 M'$. This means that $M \subseteq M'$ and that for $\mathbf{m} \in M$ and $\sigma(\mathbf{y}) \in \Sigma_1$ we have $M \models \sigma(\mathbf{m}) \quad \Leftrightarrow M' \models \sigma(\mathbf{m})$. In this case we also say that M is a 1-elementary submodel of M'. It is easy to see that

$$M \prec_1 M' \;\Leftrightarrow\; [M \models \sigma(\mathbf{m}) \Rightarrow M' \models \sigma(\mathbf{m})] \;\text{ for all } \sigma(\mathbf{y}) \in \Sigma_2.$$

A 1-elementary chain is a sequence $M_0 \prec_1 M_1 \prec_1 M_2 \prec_1 \ldots$. It is well known that the union of a 1-elementary chain is a 1-elementary extension of every model in the chain. It is worthy to note that in a 1-elementary chain the truth of Σ_2-sentences (with parameters) is preserved from left to right and the truth of Π_2-sentences (without parameters) is preserved from right to left.

By $\mathsf{Th}(M, C)$ we denote the first-order theory of M with all constants from C added to the language. This makes sense if we know how to interpret the constants of C in M.
We also recall the definition of the collection principle.

$$\mathrm{B}\Gamma := \{\forall\, x{<}t\, \exists\, y\; \varphi(x,y) \rightarrow \exists\, s\, \forall\, x{<}t\, \exists\, y{<}s\; \varphi(x,y) \mid \varphi \in \Gamma\}$$

together with a minimum of arithmetical axioms, e.g. PA^-. We now come to the actual proof of Theorem 5.

Proof of Theorem 5. Let a countable model $M \models \mathrm{PRA} + \sigma$ be given with $\sigma \in \Sigma_2$. We will construct a countable model M' of $\mathrm{I}\Sigma_1 + \sigma$ using Theorem 4.

Our strategy will be to make any $\Delta_0(\{\mathsf{Sup}_i\}_{i\in\omega})$-definable total function of M that is not bounded by any of the $m + \mathsf{Sup}_n$ $(n \in \omega,\ m \in M)$ either bounded by some $m{+}\mathsf{Sup}_n$ $(n \in \omega,\ m \in M')$ or not total in the PRA-model M'. The model M' will be the union of a Σ_1-elementary chain of models $M = M_0 \prec_1 M_1 \prec_1 M_2 \ldots \prec_1 M' = \cup_{i\in\omega} M_i$.

At each stage either the boundedness of a total $\Delta_0(\{\mathsf{Sup}_i\}_{i\in\omega})$-definable function is guaranteed (a Π_1-sentence: $\forall\, x, y\ (\varphi(x,y) \rightarrow y \leq m + \mathsf{Sup}_n(x))$) or its non-totality (a Σ_2-sentence: $\exists\, x\, \forall\, y\ \neg\varphi(x,y)$). As we shall work with a 1-elementary chain of models, functions that are dealt with need no more attention further on in the chain. Their interesting properties, that is boundedness or non-totality, are stable. By choosing the order in which functions are dealt with in a good way, eventually all total funtions of all models M_i will be considered. We shall see that as a result of this process every total function in M' that is $\Delta_0(\{\mathsf{Sup}_i\}_{i\in\omega})$-definable is bounded by some $M + \mathsf{Sup}_n$.

To properly order the functions that we shall deal with, we fix a bijective pairing function in this proof satisfying $x, y \leq \langle x, y\rangle$. We do as if the models M_n were already defined and write $f_{n0}, f_{n1}, f_{n2}, \ldots$ for the list of the (countably many) total $\Delta_0(\{\mathsf{Sup}_i\}_{i\in\omega})$-definable functions of M_n. We emphasize that we allow the functions f_{ni} to contain parameters from M_n. Furthermore we define g_n to be f_{ab} for the unique $a, b \in \omega$ such that $\langle a, b\rangle = n$.

We define $M_0 := M$.

We will define M_{n+1} to be such that g_n becomes (or remains) either bounded or non-total in it and $M_n \prec_1 M_{n+1}$. If we can do so, we are done. For suppose $M = M_0 \models \mathrm{PRA} + \sigma$. As PRA is Π_1-axiomatizable in the language containing the $\{\mathsf{Sup}_i\}_{i\in\omega}$ we get that $M' \models \mathrm{PRA}$ and likewise $M' \models \sigma$.

If now $M' \models \mathsf{Tot}(\varphi)$ for some $\varphi \in \Delta_0(\{\mathsf{Sup}_i\}_{i\in\omega})$, we see that for some n, $M_n \models \mathsf{Tot}(\varphi)$ as soon as M_n contains all the parameters that occur in φ. Thus $f = g_m$ for some $m \geq n$. Thus in M_{m+1} the function f will be surely majorized, for $M_{m+1} \models \neg\mathsf{Tot}(\varphi) \Rightarrow M' \models \neg\mathsf{Tot}(\varphi)$. Consequently $M' \models f \leq m' + \mathsf{Sup}_k$ for some $m' \in M_{m+1} \subseteq M'$, $k \in \omega$. By Theorem 4 we see that $M' \models \mathrm{I}\Sigma_1$.

If $M_n \models g_n \leq m + \mathsf{Sup}_k$ for some $m \in M_n$ and $k \in \omega$ we set $M_{n+1} := M_n$. Clearly $M_n \prec_1 M_{n+1}$ and g_n is bounded in M_{n+1} (regargless its totality).

So, suppose that g_n is total in M_n and that $M_n \models \neg(g_n \leq m + \mathsf{Sup}_k)$ for all $m \in M_n$ and all $k \in \omega$. We obtain our required model M_{n+1} in two steps.

Step 1.

We go from $M_n \prec_1 M_{n1} \models \mathsf{B}\Delta_0(\{\mathsf{Sup}_i\}_{i\in\omega})(+\mathrm{PRA})$. To this purpose, we add a fresh constant d to our language and consider

$$T := \mathsf{Th}(M_n, \{m\}_{m\in M_n}) \cup \{d > \mathsf{Sup}_k(m) \mid k \in \omega,\ m \in M_n\}.$$

As T is finitely satisfiable in M_n, we can find a countable model $M_{n0} \models T$. Let M_{n1} be the (initial) submodel of M_{n0} with domain $\{x \in M_{n0} \mid \exists k{\in}\omega\, \exists m{\in}M_n\ x \leq \mathsf{Sup}_k(m)\}$. Clearly, M_{n1} is indeed a submodel, that is, it is closed under all the Sup_k. For if $x \leq \mathsf{Sup}_l(m)$ then $\mathsf{Sup}_k(x) \leq \mathsf{Sup}_k(\mathsf{Sup}_l(m)) \leq \mathsf{Sup}_{k+l+2}(m)$. We see that M_{n1} is a model of PRA as

PRA is Π_1-axiomatized. As $M_n \subseteq M_{n1}$, we get $M_n \prec_1 M_{n1}$. For,

$$\begin{array}{lll} M_n & \models \exists x\, \varphi(x) \quad , \varphi(x) \in \Pi_1 \Rightarrow & \text{for some } m \in M_n \\ M_n & \models \varphi(m) & \Rightarrow \\ M_{n0} & \models \varphi(m) & \Rightarrow \\ M_{n1} & \models \varphi(m) & \Rightarrow \\ M_{n1} & \models \exists x\, \varphi(x). & \end{array}$$

We now see that $M_{n1} \models \mathsf{B}\Delta_0(\{\mathsf{Sup}_i\}_{i\in\omega})$. So, suppose $M_{n1} \models \forall\, x{<}t\, \exists\, y\, \varphi(x,y)$ for some $t \in M_{n1}$ and $\varphi \in \Delta_0(\{\mathsf{Sup}_i\}_{i\in\omega})$. Clearly $M_{n0} \models \forall\, x{<}t\, \exists\, y{<}d\, \varphi(x,y)$ for some $d \in M_{n0}$, actually for any $d \in M_{n0} \backslash M_{n1}$. Now by the $\Delta_0(\{\mathsf{Sup}_i\}_{i\in\omega})$ minimal number principle we get a minimal d_0 such that $M_{n0} \models \forall\, x{<}t\, \exists\, y{<}d_0\, \varphi(x,y)$. If d_0 were in $M_{n0} \setminus M_{n1}$, then $d_0 - 1$ would also suffice as a bound on the y's. The minimality of d_0 thus imposes that $d_0 \in M_{n1}$. Consequently $M_{n1} \models \exists\, d_0\, \forall\, x{<}t\, \exists\, y{<}d_0\, \varphi(x,y)$ and $M_{n1} \models \mathsf{B}\Delta_0(\{\mathsf{Sup}_i\}_{i\in\omega})$.

Step 2.

We go from[10] $M_{n1} \models \mathsf{B}\Delta_0(\{\mathsf{Sup}_i\}_{i\in\omega})(+\text{PRA})$ to a model $M_{n1} \prec_1 M_{n3} \models \text{PRA} + \neg\mathsf{Tot}(\chi_n)$. M_{n+1} will be the reduct of M_{n3} to the original language.

If $M_{n1} \models \neg\mathsf{Tot}(\chi_n)$ nothing has to be done and we take $M_{n3} = M_{n1}$. So, we assume that $M_{n1} \models \mathsf{Tot}(\chi_n)$. We consider the set

$$\Gamma := \mathsf{Th}(M_{n1}, \{m\}_{m\in M_{n1}}) \cup \{g_n(c) > m + \mathsf{Sup}_k(c) \mid m \in M_{n1},\ k \in \omega\}$$

with c a fresh constant symbol. As g_n is not majorizable in M_{n1} we see that any finite subset of Γ is satisfiable whence Γ is satisfiable. Let M_{n2} be a countable model of Γ. Of course, we can naturally embed M_{n1} in M_{n2}.

We will now see that $c > M_{n1}$. For suppose $c \leq m \in M_{n1}$. Then $M_{n1} \models \forall\, x{\leq}m\, \exists\, z\ g_n(x){=}z$.[11] By $\Delta_0(\{\mathsf{Sup}_i\}_{i\in\omega})$-collection we get $M_{n1} \models \exists\, d_0\, \forall\, x{\leq}m\, \exists\, z{\leq}d_0\ g_n(x){=}z$. But then $M_{n1} \models g_n(c) \leq d_0$ whence $M_{n1} \models \neg(g_n(c) > d_0 + \mathsf{Sup}_k(c))$. A contradiction.

Define M_{n3} to be the (initial) submodel of M_{n2} with domain $\{m \in M_{m2} \mid \exists\, k \in \omega\ \ M_{n2} \models m < \mathsf{Sup}_k(c)\}$. As $c \geq M_{n1}$ we get $M_{n1} \subseteq M_{n3}$. We now see that $M_{n1} \prec_1 M_{n3}$. For suppose $M_{n1} \models \exists\, x\ \varphi(x)$ with $\varphi(x) \in \Pi_1$ then $M_{n1} \models \varphi(m_0)$ for some $m_0 \in M_{n1}$. Consequently $M_{n2} \models \varphi(m_0)$ and as $M_{n3} \subset_e M_{n2}$ and $\varphi(m_0) \in \Pi_1$, also $M_{n3} \models \varphi(m_0)$ whence $M_{n3} \models \exists\, x\ \varphi(x)$. Clearly $M_{n3} \models \neg\mathsf{Tot}(\chi_n)$ as $g_n(c)$ can not have a value in M_{n3}. □

Corollary 3. $\forall \pi \in \Pi_2\ (\mathrm{B}\Sigma_1 \vdash \pi \Rightarrow \mathrm{PRA} \vdash \pi)$

Proof. A direct proof of this fact is given in Step 1 in the above proof.

[10] Or from the reduct of M_{n1} to the original language for that matter.
[11] We actually should substitute the $\Delta_0(\{\mathsf{Sup}_i\}_{i\in\omega})$-graph of g_n here.

5. Consistency, cuts and length of proofs

A direct consequence of the formalizability of Parsons' theorem is that PRA and $\mathrm{I}\Sigma_1$ are equi-consistent. To be more precise, for every theory T proving the totality of the superexponentiation we have that

$$T \vdash \mathsf{Con}(\mathrm{PRA}) \leftrightarrow \mathsf{Con}(\mathrm{I}\Sigma_1).$$

Consequently $\mathrm{I}\Sigma_1 \nvdash \mathsf{Con}(\mathrm{PRA})$. In this section we shall see that we can find a definable $\mathrm{I}\Sigma_1$-cut J such that $\mathrm{I}\Sigma_1 \vdash \mathsf{Con}^J(\mathrm{PRA})$. More generally, we shall show that for this cut J we actually have that for any Σ_2-sentence σ, it holds that $\mathrm{I}\Sigma_1 + \sigma \vdash \mathsf{Con}^J(\mathrm{PRA} + \sigma)$.

As in [Pud86] and [Ign90] we note that Theorem 6 implies that certain proofs in PRA must get exponentially larger than their counterparts in $\mathrm{I}\Sigma_1$. This, in a sense, says that the use of the cut-elimination, whence the super exponential blow-up, in the proof of Theorem 1 was essential.

To the best of our knowledge Ignjatovic ([Ign90]) showed for the first time that $\mathrm{I}\Sigma_1$ proves the consistency of PRA on some definable cut. His reasoning was based on a paper by Pudlák ([Pud86]). Pudlák showed in this paper by model-theoretic means that GB proves the consistency of ZF on a cut. The cut that Ignjatovic exposes is actually an RCA_0-cut. (See for example [Sim99] for a definition of RCA_0.)

The elements of the cut correspond to complexities of formulas for which a sort of truth-predicate is available. By an interpretability argument it is shown that a corresponding cut can be defined in $\mathrm{I}\Sigma_1$. It seems straightforward to generalize his result to obtain Theorem 6.

In [Joo04] an explicit $\mathrm{I}\Sigma_1$-cut J is exposed such that $\mathrm{I}\Sigma_1 \vdash \mathsf{Con}^J(\mathrm{PRA})$. Actually, a far more general result is proved there by proof theoretical methods. Namely, that for each $n{\in}\omega$ there exists some $\mathrm{I}\Sigma_n$-cut J_n such that for all Σ_{n+1}-sentences σ, $\mathrm{I}\Sigma_n + \sigma \vdash \mathsf{Con}^{J_n}(\mathrm{I}\Sigma_n^R + \sigma)$. The proof is easily formalizable in the presence of supexp.

The proof we present here is a simplification of an argument by Visser. In an unpublished note [Vis90], Visser sketched a modification of a proof of Paris and Wilkie from [WP87] to obtain our Theorem 6. Lemma 8.10 from [WP87], implies that for every $r{\in}\omega$ there is an $(\mathrm{I}\Delta_0 + \mathsf{exp})$-cut such that for every $\sigma{\in}\Sigma_2$, $\mathrm{I}\Delta_0 + \sigma + \mathsf{exp}$ proves the consistency of $\mathrm{I}\Delta_0 + \sigma + \Omega_r$ on that cut.

5.1. Basic definitions

Let us first give a definition of PRA that is useful to us in our proof. Again, we will work with the functions $\mathsf{Sup}_n(x)$ as introduced in Section 4. However, this time we will not extend our language. Rather we shall work with arithmetical definitions of the $\mathsf{Sup}_n(x)$. Let us recall the defining equations for the functions $\mathsf{Sup}_n(x)$.

- $\mathsf{Sup}_0(x) = 2{\cdot}x$
- $\mathsf{Sup}_{z+1}(0) = 1$

- $\mathsf{Sup}_{z+1}(x+1) = \mathsf{Sup}_z(\mathsf{Sup}_{z+1}(x))$

We see that $\mathsf{Sup}_z(x) = y$ can be expressed by a Σ_1-formula:[12]

$$(\mathsf{Sup}_z(x) = y) := (\exists s\, \widetilde{\mathsf{Sup}}(s, z, x, y)),$$

where $\widetilde{\mathsf{Sup}}(s, z, x, y)$ is the following Δ_0-formula:

$$\begin{array}{c}
\mathsf{Finseq}(s) \wedge \mathsf{lh}(s){=}z{+}1 \wedge \\
\mathsf{lh}(s_z){=}x{+}1 \wedge \forall\, i{\leq}z\ (\mathsf{Finseq}(s_i) \wedge [(i{<}z) \rightarrow \mathsf{lh}(s_i) = (s_{i+1})_{\mathsf{lh}(s_{i+1})-2}]) \\
\wedge\ \forall\, j{<}\mathsf{lh}(s_0)\ (s_0)_j = 2{\cdot}j\ \wedge \\
\forall\, i{<}\mathsf{lh}(s){-}1\ ((s_{i+1})_0 = 1 \wedge \forall\, j{<}\mathsf{lh}(s_{i+1}){-}1\ ((s_{i+1})_{j+1} = (s_i)_{(s_{i+1})_j})) \\
\wedge\ (s_z)_x = y.
\end{array}$$

The intuition behind the formula $\widetilde{\mathsf{Sup}}(s, z, x, y)$ is very clear. The s is a sequence of sufficiently large parts of the graphs of the $\mathsf{Sup}_{z'}$'s. Thus,

$$s = \begin{cases} [[\mathsf{Sup}_0(0), \mathsf{Sup}_0(1), \ldots, \mathsf{Sup}_0(\mathsf{lh}(s_0) - 1)], \\ \ [\mathsf{Sup}_1(0), \mathsf{Sup}_1(1), \ldots, \mathsf{Sup}_1(\mathsf{lh}(s_1) - 1)], \\ \qquad\qquad \vdots \\ \ [\mathsf{Sup}_z(0), \mathsf{Sup}_z(1), \ldots, \mathsf{Sup}_z(\mathsf{lh}(s_z) - 1)]]. \end{cases}$$

Rather weak theories already prove the main properties of the Sup_z functions (without saying anything about the definedness) like

$$\begin{array}{l}
\mathsf{Sup}_n(1) = 2, \\
\mathsf{Sup}_n(2) = 4, \\
1 \leq \mathsf{Sup}_{n+1}(y), \\
x \leq y \rightarrow \mathsf{Sup}_n(x) \leq \mathsf{Sup}_n(y), \\
(n{\leq}m \wedge x{\leq}y) \rightarrow \mathsf{Sup}_n(x){\leq}\mathsf{Sup}_m(y),
\end{array}$$

and so on.

Definition 10. PRA *is the first-order theory in the language* $\{+, \cdot, \leq, 0, 1\}$ *using only the connectives* $\neg, \rightarrow$ *and* $\forall$, *with the following* non-logical axioms.

[*A*.] *Finitely many defining* Π_1*-axioms for* $+$, $\cdot$, $\leq$, *0 and 1.*
[*B*.] *Finitely many identity axioms of complexity* Π_1.
[*C*.] *For every* $\varphi(x, \mathbf{a}){\in}\Delta_0$ *an induction axiom of complexity* Π_1 *of the form:*[13]
$\forall x\ \forall z\ (\varphi(0, z) \wedge \forall\, y{<}x\ (\varphi(y, z) \rightarrow \varphi(y{+}1, z)) \rightarrow \varphi(x, z))$.

[12] By close inspection of the defining formula we see that $\mathsf{Sup}_z(x){=}z$ can actually be regarded as a $\Delta_0(\exp)$-formula.

[13] We mean of course a Π_1-formula using only $\neg, \rightarrow$ and $\forall$, that is logically equivalent to the formula given here. By coding techniques, having just one parameter z in our induction axioms, is no real restriction. It prevents, however, getting a non-standard block of quantifiers in non-standard PRA-axioms.

[D.] *For all $z{\in}\omega$ a totality statement (of complexity Π_2) for the function* $\mathsf{Sup}_z(x)$ *in the following form:* $\forall x\,\exists s\,\exists y{\leq}s\ \widetilde{\mathsf{Sup}}(s,\overline{z},x,y)$*. Here and in the sequel $\overline{z}$ denotes the numeral corresponding to z, that is, the string* $\overbrace{1+\ldots+1}^{z \text{ times}}$.

The logical axioms *and* rules *are just as usual.*

We shall need in our proof of Theorem 6 a formalization of a proof system that has the sub-formula property. Like Paris and Wilkie we shall use a notion of tableaux proofs rather than some sequent calculus. In our discussion below we consider theories T that are formulated using only the connectives $\rightarrow$, $\neg$ and $\forall$. The other connectives will still be used as abbreviations.

Definition 11. *A* tableau proof of a contradiction *from a set of axioms T containing the identity axioms is a finite sequence $\Gamma_0, \Gamma_1, \ldots, \Gamma_r$ where the Γ_i satisfy the following conditions.*

- *For $0 \leq i \leq r$, Γ_i is a sequence of sequences of labeled formulas. The elements of Γ_i are denoted by Γ_i^j. The elements of the Γ_i^j are denoted by $\varphi_{i,j}^k(l)$ where l is the* label *of $\varphi_{i,j}^k$ and is either 0 or 1. In case $l=1$ in $\varphi_{i,j}^k(l)$, we call $\varphi_{i,j}^k$ the* active *formula of both Γ_i^j and Γ_i. Only non-atomic formulas can be active.*
- *Γ_0 contains just one finite non-empty sequence of labeled formulas. We require $\varphi_{0,0}^k{\in}T$ for $k < \mathsf{lh}(\Gamma_0^0)$.*
- *In every Γ_r^j ($j < \mathsf{lh}(\Gamma_r)$) there is an atomic formula that also occurs negated in Γ_r^j.*
- *Every $0 \leq i < r$ contains exactly one sequence Γ_i^j with an active formula in it. This sequence in its turn contains exactly one active formula.*
- *For $0 \leq i < r$, we have $\mathsf{lh}(\Gamma_i) \leq \mathsf{lh}(\Gamma_{i+1}) \leq \mathsf{lh}(\Gamma_i)+1$.*
- *For $0 \leq i < r$, we have $\mathsf{lh}(\Gamma_i^j) \leq \mathsf{lh}(\Gamma_{i+1}^j) \leq \mathsf{lh}(\Gamma_i^j)+2$.*
- *For $0 \leq i < r$, we have $\varphi_{i,j}^k = \varphi_{i+1,j}^k$ for $k < \mathsf{lh}(\Gamma_i^j)$.*
- *$\mathsf{lh}(\Gamma_i^j) < \mathsf{lh}(\Gamma_{i+1}^j)$ iff Γ_i^j contains the active formula of Γ_i. In this case, with $n = \mathsf{lh}(\Gamma_i^j)$ and $\varphi_{i,j}^m$ the active formula, one of the following holds.*[14]
  - (β) *$\varphi_{i,j}^m$ is of the form $\neg\neg\theta$ in which case $\Gamma_{i+1,j}^n = \theta$ and $\mathsf{lh}(\Gamma_{i+1}^j) = n{+}1$.*
  - (γ) *$\varphi_{i,j}^m$ is of the form $\theta_1 \rightarrow \theta_2$. In this case $\Gamma_{i+1,j}^n = \neg\theta_1$ and only in this case $\mathsf{lh}(\Gamma_{i+1}) = \mathsf{lh}(\Gamma_i)+1$. Let $p := \mathsf{lh}(\Gamma_i)$. Γ_{i+1}^p is defined as follows: $\mathsf{lh}(\Gamma_{i+1}^p) = \mathsf{lh}(\Gamma_{i+1}^j) = n+1$, $\Gamma_{i+1,p}^k = \Gamma_{i+1,j}^k$ for $k < n$ and $\Gamma_{i+1,p}^n = \theta_2$.*
  - (δ) *$\varphi_{i,j}^m$ is of the form $\neg(\theta_1 \rightarrow \theta_2)$. Only in this case $\mathsf{lh}(\Gamma_{i+1}^j) = \mathsf{lh}(\Gamma_i^j){+}2$ and $\Gamma_{i+1,j}^n = \theta_1$ and $\Gamma_{i+1,j}^{n+1} = \neg\theta_2$.*
  - (ϵ) *$\varphi_{i,j}^m$ is of the form $\forall x\ \theta(x)$. In this case $\mathsf{lh}(\Gamma_{i+1}^j) = n{+}1$ and $\Gamma_{i+1,j}^n = \theta(t)$ for some term t that is freely substitutable for x in $\theta(x)$.*

[14] We start with (β), so that we have the same labels as in Definition 8.9 from [WP87].

(ζ) *$\varphi^m_{i,j}$ is of the form $\neg\forall x\ \theta(x)$. In this case* $\mathsf{lh}(\Gamma^j_{i+1}) = n+1$ *and* $\Gamma^n_{i+1,j} = \neg\theta(y)$ *for some variable y that occurs in no formula of* Γ^j_i.

It is well-known that φ is provable from T iff there is a tableau proof of a contradiction from $T \cup \{\neg\varphi\}$. The length of tableaux proofs can grow superexponentially larger than their regular counterparts. A pleasant feature of tableaux proofs is the sub-formula property.

We will work with some suitable Δ_1-coding of assignments that are always zero on all but finitely many variables. The constant zero valuation is denoted just by 0. Also do we use well-known satisfaction predicates like $\mathsf{Sat}_{\Pi_1}(\pi, \sigma)$ for formulas $\pi \in \Pi_1$ and valuations σ. By $\mathsf{Val}(t, \sigma)$ we denote some Δ_1 valuation function for terms t and assignments σ. By $\Sigma_1(x)$ we denote the predicate that only holds on the standard model on codes of (syntactical) Σ_1-sentences.

5.2. IΣ_1 *proves the consistency of* PRA *on a cut*

Theorem 6. *There exists an* IΣ_1*-cut J such that for all $B \in \Sigma_2$ we have* I$\Sigma_1 + B \vdash \mathsf{Con}^J(\mathrm{PRA} + B)$

Proof. We will expose an IΣ_1-cut and show that I$\Sigma_1 + B \vdash \mathsf{Con}^J(\mathrm{PRA}+B)$ for any $B \in \Sigma_2$(formulated using only $\neg$, $\rightarrow$ and $\forall$). If we would have a J-proof of $\bot$ from PRA + B in I$\Sigma_1 + B$ we can also find a tableau proof of a contradiction (not necessarily in J) from $\mathrm{PRA}^J + B$, as IΣ_1 proves the totality of the superexponentiation function. By PRA^J we denote the axiom set of PRA intersected with J.

Thus, it suffices to show that I$\Sigma_1 + B \vdash \mathsf{TabCon}(\mathrm{PRA}^J + B)$. By TabCon we mean the formalization of the assertion that there is no tableau proof of a contradiction.

The cut that does the job is the following:[15]

$$J(z) := \forall\, z' {\leq} z\, \forall x\, \exists y\ \mathsf{Sup}_{z'}(x) = y.$$

First we see that $J(z)$ indeed defines a cut in IΣ_1. Obviously I$\Sigma_1 \vdash J(0)$. We now see I$\Sigma_1 \vdash J(z) \rightarrow J(z{+}1)$. For, reason in I$\Sigma_1$ and suppose $J(z)$. In order to obtain $J(z{+}1)$ it is sufficient to show that $\forall x\, \exists y\ \mathsf{Sup}_{z+1}(x) = y$. This follows from an easy Σ_1-induction. As $B \in \Sigma_2$ we may assume that $B = \exists x\ A(x)$ with $A \in \Pi_1$.

We reason in I$\Sigma_1 + B$ and assume $\neg\mathsf{TabCon}(\mathrm{PRA}^J + B)$. As B holds, for some a we have $A(a)$. We fix this a for the rest of the proof. Let $p = \Gamma_0, \Gamma_1, \ldots, \Gamma_r$ be a hypothetical tableau proof of a contradiction from $\mathrm{PRA}^J + B$.

Via some easy inductions a number of basic properties of p is established, like the sub-formula property and the fact that every Σ_1!-formula in p comes

[15] Formally speaking we should use the $\widetilde{\mathsf{Sup}}(s, z, x, y)$ predicate here.

from a PRA-axiom of the form $[D.]$, etcetera. Inductively we define for every Γ_i^j a valuation $\sigma_{i,j}$.

- $\sigma_{0,0} = 0$.
- If Γ_i^j contains no active formula, $\sigma_{i+1,j} = \sigma_{i,j}$.
- If Γ_i^j contains an active formula one of (β)-(ζ) applies. Let $m{=}\mathsf{lh}(\Gamma_i^j)$.
  (β) $\sigma_{i+1,j} = \sigma_{i,j}$.
  (γ) $\sigma_{i+1,j} = \sigma_{i+1,m} = \sigma_{i,j}$.
  (δ) $\sigma_{i+1,j} = \sigma_{i,j}$.
  (ϵ) $\sigma_{i+1,j} = \sigma_{i,j}$.
  (ζ) In this case essentially an existential quantifier is eliminated. We treat the three possible eliminations.[16]
    - The first existential quantifier in B is eliminated and B is replaced by $A(y)$. In this case $\sigma_{i+1,j} = \sigma_{i,j}$ for all variables different from y. Furthermore we define $\sigma_{i+1,j}(y) = a$.
    - The first existential quantifier in a formula of the form $\exists s\, \exists\, y{\leq}s\, \widetilde{\mathsf{Sup}}(s, \overline{z}, t, y)$ for some term t and number $z{\in}J$ is eliminated and replaced by $\exists\, y{\leq}v\, \widetilde{\mathsf{Sup}}(v, \overline{z}, t, y)$ for some variable v. In this case $\sigma_{i+1,j} = \sigma_{i,j}$ for all variables different from v. Furthermore we define $\sigma_{i+1,j}(v)$ to be the minimal number b such that
    $$\exists\, y{\leq}b\, \widetilde{\mathsf{Sup}}(b, \mathsf{Val}(\overline{z}, \sigma_{i,j}), \mathsf{Val}(t, \sigma_{i,j}), y).$$
    Note that, as $z \in J$, such a number b must exist.
    - A bounded existential quantifier in a formula of the form $\exists\, x{\leq}t\ \theta(x)$ is eliminated and $\exists\, x{\leq}t\ \theta(x)$ is replaced by $y \leq t \wedge \theta(y)$ for some variable y. In this case $\theta(y)$ is in Δ_0 (yet another induction). We define $\sigma_{i+1,j}(y)$ to be the minimal $c \leq \mathsf{Val}(t, \sigma_{i,j})$ such that $\mathsf{Sat}_{\Delta_0}(\ulcorner \theta(\overline{c}) \urcorner, \sigma_{i,j})$ if such a c exists. In case no such c exists, we define $\sigma_{i+1,j}(y) = 0$. For the other variables we have $\sigma_{i+1,j} = \sigma_{i,j}$.

It is not hard to see that $\sigma_{i,j}(x)$ has a Σ_1 or even Δ_1-graph. The proof is now completed by showing by induction on i:

$$\forall\, i{\leq}r\, \exists\, j{<}\mathsf{lh}(\Gamma_i)\, \forall\, k{<}\mathsf{lh}(\Gamma_i^j)\ (\Sigma_1(\ulcorner \varphi_{i,j}^k \urcorner) \rightarrow \mathsf{Sat}_{\Sigma_1}(\ulcorner \varphi_{i,j}^k \urcorner, \sigma_{i,j})). \quad (\dagger)$$

Note that the statement is indeed Σ_1 as in $\mathrm{I}\Sigma_1$ we have the Σ_1-collection principle which tells us that the bounded universal quantifiers can be somehow pushed inside the unbounded existential quantifier of the Sat_{Σ_1}.

Once we have shown $(\dagger)$, we have indeed finished the proof as every Γ_r^j $(j{<}\mathsf{lh}(\Gamma_r))$ contains some atomic formula and its negation. Atomic formulas are certainly Σ_1 which gives for some $j{<}\mathsf{lh}(\Gamma_r)$ and some atomic formula θ, both $\mathsf{Sat}_{\Sigma_1}(\ulcorner \theta \urcorner, \sigma_{r,j})$ and $\mathsf{Sat}_{\Sigma_1}(\ulcorner \neg\theta \urcorner, \sigma_{r,j})$ and we have arrived at a contradiction. Hence $\mathsf{TabCon}(\mathrm{PRA}^J + B)$.

[16] Again, to see (in $\mathrm{I}\Sigma_1$) that these are the only three possibilities, an induction is executed.

As announced (†) will be proved by induction on i. If i=0, as there are no Σ_1-formulas in Γ_0^0, (†) holds in a trivial way.

For the inductive step, let $i<r$ and $j<\mathsf{lh}(\Gamma_i)$ such that

$$\forall\, k<\mathsf{lh}(\Gamma_i^j)\ (\Sigma_1(\ulcorner\varphi_{i,j}^k\urcorner) \rightarrow \mathsf{Sat}_{\Sigma_1}(\ulcorner\varphi_{i,j}^k\urcorner, \sigma_{i,j})).$$

We look for $j'<\mathsf{lh}(\Gamma_{i+1})$ such that

$$\forall\, k<\mathsf{lh}(\Gamma_{i+1}^{j'})\ (\Sigma_1(\ulcorner\varphi_{i+1,j'}^k\urcorner) \rightarrow \mathsf{Sat}_{\Sigma_1}(\ulcorner\varphi_{i+1,j'}^k\urcorner, \sigma_{i+1,j'})). \quad (\ddagger)$$

If Γ_i^j contains no active formula, $\Gamma_{i+1}^j=\Gamma_i^j$ and $\sigma_{i+1,j}=\sigma_{i,j}$, and we can just take $j'=j$.

So, we may assume that Γ_i^j contains an active formula, say $\varphi_{i,j}^m$, and one of (β)-(ζ) holds. In the cases (β), (γ) and (δ) it is clear which j' should be taken such that (‡) holds. We now concentrate on the two remaining cases.

(ζ). Here $\varphi_{i,j}^m$ is of the form $\exists x\ \theta(x)$. We only need to consider the case that $\exists x\ \theta(x) \in \Sigma_1$. By an easy induction we see that $\exists x\ \theta(x)$ is either Δ_0 or a subformula (modulo substitution of terms) of an axiom of PRA from group [D].

In case $\varphi_{i,j}^m = \exists x\ \theta(x)$ and $\exists x\ \theta(x) \in \Delta_0$, for some $v \notin \Gamma_i^j$, $\varphi_{i+1,j}^m = \theta(v)$. As we know that $\mathsf{Sat}_{\Sigma_1}(\ulcorner\varphi_{i,j}^m\urcorner, \sigma_{i,j})$, we see that $\sigma_{i+1,j}$ is tailored such that $\mathsf{Sat}_{\Delta_0}(\ulcorner\varphi_{i+1,j}^m\urcorner, \sigma_{i+1,j})$ holds. Clearly also $\mathsf{Sat}_{\Sigma_1}(\ulcorner\varphi_{i+1,j}^m\urcorner, \sigma_{i+1,j})$ and we can take j=j' to obtain (‡).

The other possibility is $\varphi_{i,j}^m = \exists s\, \exists\, y{\leq}s\ \widetilde{\mathsf{Sup}}(s, \overline{z}, t, y)$ for some (possibly non-standard) term t. Consequently $\varphi_{i+1,j}^m = \exists\, y{\leq}v\ \widetilde{\mathsf{Sup}}(v, \overline{z}, t, y)$ for some $v \notin \Gamma_i^j$. Again $\sigma_{i+1,j}$ is tailored such that $\mathsf{Sat}_{\Delta_0}(\ulcorner\varphi_{i+1,j}^m\urcorner, \sigma_{i+1,j})$ holds and we can take j=j' to obtain (‡).

(ϵ). We only need to consider the case $\varphi_{i,j}^m = \forall x\ \theta(x)$ with $\theta(x) \in \Sigma_1$. In case $\forall x\ \theta(x) \in \Sigma_1$, the induction hypothesis and the definition of $\sigma_{i+1,j}$ guarantees us that j=j' yields a solution of (‡). So, we may assume that $\forall x\ \theta(x) \notin \Sigma_1$. By an easy induction we see that thus $\forall x\ \theta(x)$ is $A(a)$ or $\theta(x)$ has one of the following forms:

1. A subformula (modulo substitution of terms) of an axiom of PRA of the form [A] or [B],
2. A subformula (modulo substitution of terms) of an induction axiom [C],
3. $\exists s\, \exists\, y{\leq}s\ \widetilde{\mathsf{Sup}}(s, \overline{z}, t, y)$ for some (possibly non-standard) term t and some $z{\in}J$.

Our strategy in all cases but 3 will be to show that[17]

$$\forall\sigma\ \mathsf{Sat}_{\Pi_1}(\ulcorner\forall x\ \theta(x)\urcorner, \sigma). \quad ♣$$

[17] $\forall\sigma\ \mathsf{Sat}_{\Pi_1}(\ulcorner\varphi\urcorner, \sigma)$ is often denoted by $\mathsf{True}_{\Pi_1}(\varphi)$.

This is sufficient as

$$\begin{array}{ll}\forall\sigma\ \mathsf{Sat}_{\Pi_1}(\ulcorner\forall x\ \theta(x)\urcorner,\sigma) & \Rightarrow\\ \forall\sigma\,\forall x\ \mathsf{Sat}_{\Delta_0}(\ulcorner\theta(v)\urcorner,\sigma[v/x]) & \Rightarrow\\ \forall\sigma'\ \mathsf{Sat}_{\Delta_0}(\ulcorner\theta(v)\urcorner,\sigma') & \Rightarrow\\ \forall\sigma\ \mathsf{Sat}_{\Delta_0}(\ulcorner\theta(t)\urcorner,\sigma) & \Rightarrow\\ \forall\sigma\ \mathsf{Sat}_{\Sigma_1}(\ulcorner\theta(t)\urcorner,\sigma). & \end{array}$$

Here v is some fresh variable, $\theta[v/x]$ denotes the formula where x is substituted for v in $\theta(v)$, and $\sigma[v/x]$ denotes the valuation which (possibly) only differs from σ in that it assigns to the variable v the value x.

The strategy to prove 3 is quite similar. The formula $\forall x\,\exists s\,\exists\, y{\leq}s\ \widetilde{\mathsf{Sup}}(s,z,x,y)$ is a standard formula that holds if $z\in J$, whence for some variable v we have

$$\forall\sigma\ \mathsf{Sat}_{\Pi_2}(\ulcorner\forall x\,\exists s\,\exists\, y{\leq}s\ \widetilde{\mathsf{Sup}}(s,v,x,y)\urcorner,\sigma[v/z])$$

and thus also

$$\forall\sigma\ \mathsf{Sat}_{\Pi_2}(\ulcorner\forall x\,\exists s\,\exists\, y{\leq}s\ \widetilde{\mathsf{Sup}}(s,\overline{z},x,y)\urcorner,\sigma).$$

We immediately see that

$$\forall\sigma\ \mathsf{Sat}_{\Sigma_1}(\ulcorner\exists s\,\exists\, y{\leq}s\ \widetilde{\mathsf{Sup}}(s,\overline{z},t,y)\urcorner,\sigma).$$

The proof is thus finished if we have shown ♣ in case $\forall x\ \theta(x)$ is either $A(a)$ or a subformula of an axiom of the groups $[A]$, $[B]$ and $[C]$. The only hard case is whenever $\forall x\ \theta(x)$ is a subformula of a PRA axiom of group $[C]$, as the other cases concern true standard Π_1-sentences only. By an easy induction we see that it is sufficient to show that for every $\varphi\in\Delta_0$

$$\forall x\ \mathsf{Sat}_{\Pi_1}(\ulcorner\forall z\ (\varphi(0,z)\wedge\forall\, y{<}v\ (\varphi(y,z)\to\varphi(y{+}1,z))\to\varphi(v,z))\urcorner,\sigma_{0,0}[v/x]).$$

This is proved by a Π_1-induction on x. Note that in $\mathrm{I}\Sigma_1$ we have indeed access to Π_1-induction as $\mathrm{I}\Sigma_1\equiv\mathrm{I}\Pi_1$. The fact that φ can be non-standard urges us to be very precise.

If $x{=}0$ we are done if we have shown

$$\mathsf{Sat}_{\Pi_1}(\ulcorner\forall z\ (\varphi(0,z)\wedge\forall\, y{<}0\ (\varphi(y,z)\to\varphi(y+1,z))\to\varphi(0,z))\urcorner,\sigma_{0,0})$$

or equivalently

$$\forall z\ \mathsf{Sat}_{\Delta_0}(\ulcorner\varphi(0,w)\to\varphi(0,w)\urcorner,\sigma_{0,0}[w/z]).$$

By an easy induction on the length of φ we can show that for any σ

$$\mathsf{Sat}_{\Delta_0}(\ulcorner\varphi(0,w)\to\varphi(0,w)\urcorner,\sigma).$$

For the inductive step we have to show

$$\mathsf{Sat}_{\Pi_1}(\ulcorner\forall z\ (\varphi(0,z)\wedge\forall\, y{<}v\ (\varphi(y,z)\to\varphi(y{+}1,z))\to\varphi(v,z))\urcorner,\sigma_{0,0}[v/x{+}1])$$

or equivalently that for arbitrary[18] z

$$\mathsf{Sat}_{\Delta_0}(\ulcorner\varphi(0,w)\wedge\forall\, y{<}v\ (\varphi(y,w)\rightarrow\varphi(y{+}1,w))\rightarrow\varphi(v,w)\urcorner,\sigma_{0,0}[v/x{+}1][w/z]).$$

The reasoning by which we obtain this, is almost like φ were standard. So, we suppose

$$\mathsf{Sat}_{\Delta_0}(\ulcorner\varphi(0,w)\wedge\forall\, y{<}v\ (\varphi(y,w)\rightarrow\varphi(y+1,w))\urcorner,\sigma_{0,0}[v/x+1][w/z]) \qquad (\natural)$$

and set out to prove

$$\mathsf{Sat}_{\Delta_0}(\ulcorner\varphi(v,w)\urcorner,\sigma_{0,0}[v/x+1][w/z]).$$

The induction hypothesis together with some basic properties of the Sat predicates gives us

$$\mathsf{Sat}_{\Delta_0}(\ulcorner\varphi(0,w)\wedge\forall\, y{<}v\ (\varphi(y,w)\rightarrow\varphi(y{+}1,w))\rightarrow\varphi(v,w)\urcorner,\sigma_{0,0}[v/x][w/z]). \quad (\sharp)$$

A witnessing sequence for ($\natural$) is also a witnessing sequence for

$$\mathsf{Sat}_{\Delta_0}(\ulcorner\varphi(0,w)\wedge\forall\, y{<}v\ (\varphi(y,w)\rightarrow\varphi(y+1,w))\urcorner,\sigma_{0,0}[v/x][w/z]).$$

Combining this with ($\sharp$) gives us $\mathsf{Sat}_{\Delta_0}(\ulcorner\varphi(v,w)\urcorner,\sigma_{0,0}[v/x][w/z])$. Also from ($\natural$) we get $\mathsf{Sat}_{\Delta_0}(\ulcorner\varphi(v,w)\rightarrow\varphi(v+1,w)\urcorner,\sigma_{0,0}[v/x][w/z])$, so that we may conclude $\mathsf{Sat}_{\Delta_0}(\ulcorner\varphi(v{+}1,w)\urcorner,\sigma_{0,0}[v/x][w/z])$. A witnessing sequence for the latter is also a witnessing sequence for

$$\mathsf{Sat}_{\Delta_0}(\ulcorner\varphi(v,w)\urcorner,\sigma_{0,0}[v/x+1][w/z]).$$

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[18] By $\sigma[v/x][w/z]$ we mean sequential substitution. This is not an important detail, as we may assume that we have chosen v and w such that no variable clashes occur.

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