Midterm exam, The Logic of Provability

The exam is from 14:00-17:00.
You are allowed to use the book and the notes from the internet.
Write your name and student number on every paper you hand in.

1. In this exercise we will reason in PA and show the commutativity of multiplication, that is $x \cdot y = y \cdot x$. Throughout the exercise we are allowed to use the commutativity and associativity of addition. (They are expressed by Lemmata (2) and (3) of Chapter 2.)

    (a) Prove that $\forall y \; 0 \cdot y = 0$.

    (b) Consider the proof of Lemma (2) of Chapter 2 and substract a proof of $x + \mathbf{S}y = \mathbf{S}x + y$ from it.

    (c) Prove that $\forall y \; \mathbf{S}x \cdot y = x \cdot y + y$.

    (d) Prove that $x \cdot y = y \cdot x$.

2. (a) Calculate the Gödel number of $\perp \to \perp$.

    (b) Find two non-equivalent fixed points of the formula $x = \ulcorner \perp \to \perp \urcorner$. (Hint: do *not* use the fixed point operator from the book.)

3. Gödel proved his incompleteness theorems by considering a fixed point G of $\neg \mathsf{Bew}_{\mathsf{T}}(x)$, that is, a sentence such that $T \vdash G \leftrightarrow \neg \mathsf{Bew}_{\mathsf{T}}(G)$.

    (a) Formulate Gödel's first incompleteness theorem for PA.

    (b) Formulate Gödel's second incompleteness theorem for PA.

    (c) Show how the second incompleteness theorem follows from the first incompleteness theorem together with the arithmetical soundness of $\mathbf{K4}$ and Theorem 24(a.) of Chapter 1.

4. (a) Exhibit proofs to show that $\mathbf{K} \vdash (\Box\phi \vee \Box\psi) \to \Box(\phi \vee \psi)$ and $\mathbf{K4} \vdash (\Box\phi \wedge \Box\Box\psi) \to \Box\Box(\phi \wedge \psi)$.

    (b) Show for each of the above implications that the direction can not be reversed in $\mathbf{GL}$.

5. (a) We quote from the handout on the modal completeness of $\mathbf{GL}$:

    > We note that changing the order in which you eliminate problems can yield different counter models. The "minimal countermodel" is obtained by always first eliminating problems at the highest level.

    Explain (do not prove) the phenomenon that eliminating problems at the highest level first will certainly not yield a model with more nodes than the model that occurs from the step-by-step method when problems at the lowest level are first eliminated.

    (b) Prove that $\mathbf{GL} \nvdash \Box\Box A \vee \Box\Diamond\neg A$.