Exercises week 1.

1. We say that a finite sequence σ *builds up*, or *defines* a term t, if the last element of σ is t and each element in σ is either 0, a variable, or built up by means of $+$ or $\times$ with two elements that occured earlier in the sequence, or built up by means of **S**and an element that occured earlier in the sequence.

   Give two different defining sequences of $(\mathbf{SS}0 + v_3) + \mathbf{S}v_4$. (Two sequences are equal if and only if all there elements and the order in which they occur are equal.)

   Give two different defining sequences of $\mathbf{SSS}0$.

2. How is $(\mathbf{SS}0 + v_3) + \mathbf{S}v_4$ officially written in the language of PA?

3. Rewrite axiom 6 of PA in the official language of PA.

4. Provide a proof of $A \to ((A \to \bot) \to B)$ in the system **Hi** of the handout.

5. Read the definition of a proof in PA that is given on page 19 of Boolos carefully. Show that if PA $\vdash A$ and PA $\vdash B$, that then PA $\vdash A \wedge B$.

6. Show that every term has infinitely many different defining sequences.

7. Give a formula using only the logical connectives $\to, \bot$ that is classically equivalent to $(p \wedge q) \vee r$. Show that $\to, \bot$ is truth functionally complete for propositional (classical) logic. Show that we can express the existential quantifier using the universal quantifier and negation.

8. Give a proof in natural deduction with identity of $\forall x, y, z\ (x \neq y \to x \neq z \vee y \neq z)$.

9. Write $(x + S0) \cdot (x + S0) + SS0$ as a triple conform our convention on what terms are and how we represent them. (In this exercise, x is a *metavariable*.)

10. Formulate precisely what is meant by unique readability on page 18.

11. Why was Gödel's incompleteness theorem considerd to be a death-blow to Hilbert's programme?

12. Proof the deduction theorem for Hilbert style proofs for predicate logic, i.e. $\Gamma, A \vdash_H B \Leftrightarrow \Gamma \vdash_H A \to B$. Provide a proof in Hilbert style of $A \wedge B \to B \wedge A$. Study the transformation of proofs in natural deduction into Hilbert style proofs as is given in "Basic Proof Theory" of Troelstra and Schwichtenberg and provide an upperbound on the growth of the length of related proofs.