Final exam
The Logic of Provability and Interpretability

The exam is from 13:00-16:00.
You are allowed to use the book and the notes from the internet.
Write your name and student number on every paper you hand in.

1. Decide for each of the following formulas if they are provable in **GL** or not. Motivate your answer.

   (a) $\Box(A \to B) \vee \Box(B \to A)$

   (b) $\Box(\Box A \to \Box B) \vee \Box(\Box B \to \Box A)$

   (c) $\Box(\Box A \to A) \to \Box(\Box A \wedge A)$

   (d) $\Box(A \to (\Box B \to B)) \to \Box(A \to B)$

   (e) $\Diamond(p \wedge \neg q) \to \Diamond(p \wedge \neg q \wedge \Box(p \to q))$

2. (a) Write down a Δ_0-formula $\mathsf{Divides}(a, b)$, that holds on a and b iff (if and only if) a divides b. We will also write $a|b$.

   (b) Write down a Δ_0-formula $\mathsf{Prime}(a)$, that holds on a iff a is a prime number. (You may use the abbreviation that denotes the formula you have just defined in 2a .)

   (c) Goldbach's conjecture states that every even number larger than two can be written as the sum of two prime numbers. Express Goldbach's conjecture by a Π (Π^0_1) sentence.

   (d) If PA does not prove the negation of goldbach's conjecture, then it must actually be true. Explain this.

3. (a) Prove that $\mathbf{IL} \vdash \Box A \leftrightarrow \neg A \rhd \bot$.

   (b) Rewrite Löb's axiom to an equivalent formula only containing the $\rhd$ symbol.

4. In this exercise G_1 and G_2 are two (different) fixed points of $\neg\mathsf{Bew}(x)$, that is, $\mathrm{PA} \vdash G_1 \leftrightarrow \neg\mathsf{Bew}(\ulcorner G_1 \urcorner)$ and $\mathrm{PA} \vdash G_2 \leftrightarrow \neg\mathsf{Bew}(\ulcorner G_2 \urcorner)$.

   (a) Give an elementary proof of the fact that $\mathrm{PA} \vdash \mathsf{Con}(\top) \to \neg\mathsf{Bew}(\ulcorner G_1 \urcorner)$. (Hint: Try to prove the contraposition by applying provable Σ_1-completeness.)

   (b) Prove that $\mathrm{PA} \vdash G_1 \leftrightarrow G_2$. (You may use theorems from the book.)

   (c) Now use the fact that actually $\mathrm{PA} \vdash G_1 \leftrightarrow \mathsf{Con}(\top)$ to show that $\mathrm{PA} \nvdash \mathsf{Con}(\top) \to \neg\mathsf{Bew}(\ulcorner \neg G_1 \urcorner)$. (We would need here ω-consistency or 1-consistency.)

5. (a) Show that $\mathbf{IL} \vdash A \rhd \Diamond B \to \Diamond A \rhd B$.

   (b) Show that $\mathbf{IL} \nvdash (A \rhd B) \vee (B \rhd A)$.

(c) Show that $\mathbf{ILM} \vdash A \rhd \Diamond B \to \Box(A \to \Diamond B)$.

6. This exercise concerns the iteration of consistency statements. We define a sequence of sentences inductively as follows.

$$\begin{aligned}&\mathsf{Con}^0(\top) := \top\\&\mathsf{Con}^{n+1}(\top) := \mathsf{Con}_{\mathrm{PA}}(\mathsf{Con}^n(\top)) \quad (= \neg\mathsf{Bew}_{\mathrm{PA}}(\ulcorner\neg\mathsf{Con}^n(\top)\urcorner))\end{aligned}$$

Furthermore, we define a sequence of theories inductively as follows.

$$\begin{aligned}&\mathsf{Cons}^0(\mathrm{PA}) := \mathrm{PA}\\&\mathsf{Cons}^{n+1}(\mathrm{PA}) := \mathsf{Cons}^n(\mathrm{PA}) + \mathsf{Con}_{\mathsf{Cons}^n(\mathrm{PA})}(\top)\end{aligned}$$

(Where $\mathsf{Con}_{\mathsf{Cons}^n(\mathrm{PA})}(\top)$ is of course the sentece $\neg\mathsf{Bew}_{\mathsf{Cons}^n(\mathrm{PA})}(\ulcorner\bot\urcorner)$.)

(a) Show that $\mathrm{PA} \vdash \Box_{\mathrm{PA}+\alpha}\beta \leftrightarrow \Box_{\mathrm{PA}}(\alpha \to \beta)$. (We have omitted the quotes $\ulcorner\urcorner$ here.)

(b) Use 6a to show that $\mathrm{PA} \vdash \Diamond_{\mathrm{PA}+\alpha}\top \leftrightarrow \Diamond_{\mathrm{PA}}\alpha$.

(c) Apply the definition to write down the expression for $\mathsf{Cons}^2(\mathrm{PA})$.

(d) We call two theories U and V extensionally equivalent, we write $U \Leftrightarrow V$, iff they have the same set of theorems.

Prove by induction on n that

$$\mathrm{PA} + \mathsf{Con}^n(\top) \Leftrightarrow \mathsf{Cons}^n(\mathrm{PA}).$$