Transfinite reflection principles and subsystems of second-order arithmetic

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- Present reflection principles in second-order arithmetic and show how ACA_0 can also be represented as reflection over RCA_0.
- Show how infinitary reflection principles may also be used to represent more of the 'Big Five' systems of Reverse Mathematics.

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We assume all theories are elementarily presented:

$$T \vdash \phi \quad \Leftrightarrow \quad \exists x \ \underbrace{Proof_T(x, \overline{\ulcorner \phi \urcorner})}_{\in \Delta^0_0}.$$

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- $\top := \; 0 = 0$
- $\bot := \; \neg \top$
- $Cons[T] := \Diamond_T \top \in \Pi^0_1$

Some first-order theories of arithmetic

Induction schema:

$$I\phi = \phi(0) \wedge \forall x \big(\phi(x) \to \phi(x+1)\big) \to \forall x \phi(x).$$

$$I\Gamma = \{I\phi : \phi \in \Gamma\}.$$

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- Peano arithmetic:

$$\text{PA} := \text{Q} + I\Pi_\omega^0$$

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If Γ is a set of formulas,

$$RFN_\Gamma[T] := \{RFN_\phi[T] : \phi \in \Gamma\}.$$

Extending theories by reflection

Löb's rule: T only proves its reflection instances when we already have that $T \vdash \phi$:

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This generalizes Gödel's second incompleteness theorem if $\phi = \bot$:

$$\frac{T \vdash \Diamond_T \top}{T \vdash \bot}.$$

Arithmetic through reflection

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More specifically:

Theorem (Beklemishev)

For all $n \geq 1$, $I\Sigma_n \equiv \mathrm{EA} + \mathit{RFN}_{\Sigma_{n+1}}[\mathrm{EA}]$.

This was previously proven for $n \geq 2$ by Leviant using PRA.

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- By reflection we have $I\phi$.

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Solution: Cut elimination!

The Tait calculus

Sequent-based calculus, where all negations are pushed down to atomic formulas.

$$(\text{LEM}) \qquad \frac{}{\Gamma, \alpha, \neg\alpha}$$

$$(\wedge) \qquad \frac{\Gamma, \phi \qquad \Gamma, \psi}{\Gamma, \phi \wedge \psi} \qquad\qquad (\vee) \qquad \frac{\Gamma, \phi, \psi}{\Gamma, \phi \vee \psi}$$

$$(\forall) \qquad \frac{\Gamma, \phi(v)}{\Gamma, \forall x \phi(x)} \qquad\qquad (\exists) \qquad \frac{\Gamma, \phi(t)}{\Gamma, \exists x \phi(x)}$$

$$(\text{CUT}) \qquad \frac{\Gamma, \phi \qquad \Gamma, \neg\phi}{\Gamma},$$

where α is atomic and v does not appear free in Γ.

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In fact, we do not need full PA.

Let EA^+ be the theory $\mathrm{EA}+$"the superexponential is total".

Then, EA^+ suffices to prove cut-elimination.

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- Since all axioms of EA are provable in PA, we conclude that ϕ.

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Formally,

$$[n]_T\phi \equiv \exists\psi\big(True_{\Pi^0_n}(\psi) \wedge \Box_T(\psi \to \phi)\big) \in \Sigma^0_{n+1}.$$

Reflection and *n*-consistency

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Corollary

$\mathrm{PA} \equiv \mathrm{EA} + \{\langle \bar{n}\rangle_{\mathrm{EA}}\top : n < \omega\}.$

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- Π^1_n, Σ^1_n formulas have n alternating second-order quantifiers.

Basic second-order axioms

Induction axiom (*Ind*):

$$\forall X\Big(0 \in X \wedge \forall n\,(n \in X \to (n+1) \in X) \to \forall n\,(n \in X)\Big)$$

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$CA(\Gamma)$: comprehension for $\phi \in \Gamma$.

$CA(\Delta^0_1)$: for $\pi \in \Pi^0_1$, $\sigma \in \Sigma^0_1$,

$$\forall n\big(\pi(n) \leftrightarrow \sigma(n)\big) \to \exists X \forall n\,\big(n \in X \leftrightarrow \sigma(n)\big).$$

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- ECA_0: $\mathrm{Q} + \mathit{CA}(\Delta^0_0) + \mathit{Ind} + \mathit{exp}$.
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Goal: Represent ACA_0 in the form $\mathrm{RCA}_0 + R$ where R is some appropriate reflection principle.

First approximation:

Let us first consider the theory

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However, we do not get any new comprehension.

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Note: $\langle 0\rangle_T\top$ implies $\exists P\ Thm_T(P)$.

Our second approximation

For a set of formulas Γ, we define the schema

$$0\text{-}RFN_\Gamma[T] = \forall X \, \forall n \big([0]_T \phi(\bar{n}) \to \phi(n) \big).$$

(for $\phi \in \Gamma$).

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Our second approximation is

$$\mathrm{RCA}_0 + 0\text{-}RFN_{\Sigma^0_1}[\mathrm{RCA}_0].$$

Does 0-$RFN_{\Sigma^0_1}[\mathrm{RCA}_0]$ prove ACA_0?

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Reasoning in $\mathrm{RCA}_0 + 0\text{-}RFN_{\Sigma^0_1}[\mathrm{RCA}_0]$, there exists P such that $Thm_{\mathrm{RCA}_0}(P)$ holds. We instead form the set

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- But we lose completeness when ϕ has free set variables!

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Theorem

$\mathrm{ACA}_0 \equiv \mathrm{RCA}_0 + 0\text{-}\mathit{OrRFN}_{\Pi^1_2}[\mathrm{RCA}_0]$.

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- Then we may use ω-models of RCA_0 to prove reflection.

ω-models

Definition

An ω-model is a set $\mathfrak{M} = (\mathfrak{M}_n)_{n\in\mathbb{N}}$ of subsets of $\mathbb{N}$. We write $\mathfrak{M} \models \phi$ if ϕ holds when all first-order quantifiers range over $\mathbb{N}$ and all second-order quantifiers over $\{\mathfrak{M}_n\}_{n\in\mathbb{N}}$

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A partial satisfaction class on $\mathfrak{M}$ for Γ is a set S such that for all $\phi \in \Gamma$, $\phi \in S$ if and only if $\mathfrak{M} \models \phi$.

ω-models of RCA_0

Theorem

ACA_0 *proves that, given any set* X*, there is an* ω*-model* $\mathfrak{M}$ *of* RCA_0 *such that* $\mathfrak{M} \models \mathsf{RCA}_0$ *and* $\mathfrak{M}_0 = X$*.*

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Theorem

Given a finite set of formulas Γ*,* ACA_0 *proves that, given an* ω*-model* $\mathfrak{M}$*, there is a partial satisfaction class* S *for the set of substitution instances of formulas of* Γ *and their subformulas.*

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- By upwards-persistence of Σ^1_1 formulas, $\exists Y \psi(n, X, Y, Z)$ holds in $\mathbb{N}$.
- But X was arbitrary, so $\forall X \exists Y \psi(n, X, Y, Z)$ holds in $\mathbb{N}$.

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is a rule of T and for all $i \leq k$ we have that $[m]_P\Delta_i$ then also $[m]_P\Gamma$;

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3. if $i < j \leq n$ and for all k, $[i]_P(\Gamma, \phi(\bar{k}))$, then $[j]_P(\Gamma, \forall x\ \phi(x))$.

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Let $IPC^n_T(P)$ be a formula expressing that P is an iterated provability class of depth n for T.

Reasoning about strong provability operators in ACA_0

Theorem

Given $n > 0$ and a theory T, it is provable in ACA_0 that an iterated provability class of depth n exists for T.

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If cuts are not allowed we will write $[n|X]^{cf}_T\Gamma$ (although then we may also add negated axioms to Γ).

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2. for all $n, m \in |\Lambda|$, $n <_\Lambda m$ if and only if $\sigma(n, m)$.

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What if we allow cuts?

Arithmetic Transfinite Recursion

Define

$$TR_{\phi}^{\Lambda}(X, Y) \equiv \forall n \forall \lambda \big(\langle n, \lambda \rangle \in Y \leftrightarrow \phi(n, \lambda, X, Y_{<_{\Lambda}\lambda})\big).$$

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The second-order system ATR_0 is ACA_0 with the axiom scheme

$$\forall X \forall \Lambda \Big(wo(\Lambda) \to \exists Y\, TR_\phi^\Lambda(X, Y)\Big),$$

where the formula ϕ is arithmetic.

Arithmetic Transfinite Recursion

X

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X $\quad$ Y_0

$$Y_0 = \{n \in \mathbb{N} : \phi(n, 1, X, \varnothing)\}$$

Arithmetic Transfinite Recursion

$X \qquad \underbrace{Y_0}_{Y_{<_\Lambda 1}} \qquad Y_1$

$$Y_1 = \{n \in \mathbb{N} : \phi(n, 1, X, Y_{<_\Lambda 1})\}$$

Arithmetic Transfinite Recursion

$$X \qquad \underbrace{Y_0 \qquad Y_1}_{Y_{<_\Lambda 2}} \qquad Y_2$$

$$Y_2 = \{n \in \mathbb{N} : \phi(n, 1, X, Y_{<_\Lambda 2})\}$$

Arithmetic Transfinite Recursion

$$X \qquad \underbrace{Y_0 \qquad Y_1 \qquad Y_2 \ldots}_{Y_{<_\Lambda \omega}} \qquad Y_\omega$$

$$Y_\omega = \{n \in \mathbb{N} : \phi(n, 1, X, Y_{<_\Lambda \omega})\}$$

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$$X \qquad \underbrace{Y_0 \qquad Y_1 \qquad Y_2 \ldots \qquad Y_\omega}_{Y_{<_\Lambda \omega+1}} \quad Y_{\omega+1}$$

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Goal: Use strong reflection principles to represent ATR_0.

Predicative reflection principles

Predicative consistency:

$$PredCons[T] := \forall \Lambda \forall X (wo(\Lambda) \rightarrow \langle \Lambda | X \rangle_T \top)$$

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$$PredRFN_\Gamma[T] := \{PredRFN_\phi[T] : \phi \in \Gamma\}$$

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$$PredCons[T] := \forall \Lambda \forall X (wo(\Lambda) \to \langle \Lambda | X \rangle_T \top)$$

Predicative reflection:

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Given a well-order Λ and arithmetic $\phi(n, X) \in \Pi^0_{2m}$, we wish to construct a set R satisfying

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The set R satisfies $TR^\Lambda_\phi(R)$ by completeness and reflection.

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Impredicativity: The set Y is defined using a collection which includes Y!

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- As before, $[\infty|X]_T\Gamma$ means that we also have an oracle for X.

Existence of an *SPC*

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It is provable in Π^1_1*-CA that given a theory* T *and a set* X *there exists an SPC for* T *with and oracle for* X.

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- By definition, P is contained in any set Y satisfying $C(Y)$, hence it is the least such set. $\square$

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Corollary

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β-models and Π^1_1 comprehension

Definition

A β-model is a full ω-model $\mathfrak{M}$ such that whenever $\phi \in \Pi^1_1$ is a formula with parameters in $\mathfrak{M}$ such that $\mathfrak{M} \models \phi$, it follows that $\mathbb{N} \models \phi$.

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It is provable in Π^1_1-CA_0 *that any* β*-model is a model of* ATR_0*, or even the stronger* Π^1_ω-TI_0*.*

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- Can these principles be used for Π^0_1 ordinal analysis in the spirit of Beklemishev's analysis of PA?

Thank you!