An escape from Vardanyan's Theorem

Ana de Almeida Borges Joost J. Joosten

Universitat de Barcelona

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In this talk we...

- Discuss known shortcomings of quantified provability logic
- Introduce QRC$_1$ as a solution
- State obtained results about QRC$_1$
- Sketch a couple of proofs

Provability Logics

- Interpret $\Box$ as "is provable in a (specific) formal theory"
- Interpret $\Diamond$ as "is consistent with that formal theory"

Examples:

- GL is K4 $+ \Box(\Box\varphi \to \varphi) \to \Box\varphi$ (Löb's axiom)
- GLP is a polymodal version of GL, with $[0], [1], \ldots$ as modalities
  - Decidability is PSPACE-complete
- RC is the strictly positive fragment of GLP, with statements of the form $\varphi \vdash \psi$, where φ, ψ are in the language built from $\top$, p, $\wedge$, $\langle 0 \rangle, \langle 1 \rangle, \ldots$
  - E.g. $\langle 1 \rangle p \vdash \langle 0 \rangle p$
  - Decidability is in PTIME

Arithmetical realizations

It is possible to express Gödel's provability predicate in PA:

$$\mathsf{Prov}_{\mathsf{PA}}(\varphi) := \exists p\, \mathsf{Proof}_{\mathsf{PA}}(p, \varphi)$$

Let $\mathcal{L}_\Box$ be the language of GL.
An arithmetical realization is any function $(\cdot)^\star$ taking:

$$\begin{aligned} \text{formulas in } \mathcal{L}_\Box &\to \text{sentences in } \mathcal{L}_{\mathsf{PA}} \\ \text{propositional variables} &\to \text{arithmetical sentences} \\ \text{boolean connectives} &\to \text{boolean connectives} \\ \Box &\to \mathsf{Prov}_{\mathsf{PA}} \end{aligned}$$

Solovay's Theorem

Theorem (Solovay, 1976)

Let $\varphi \in \mathcal{L}_\Box$. *Then:*

$$\mathsf{GL} \vdash \varphi$$
$$\Updownarrow$$
$$\mathsf{PA} \vdash (\varphi)^\star \text{ for any arithmetical realization } (\cdot)^\star$$

This can be written as:

$$\mathsf{GL} = \{\varphi \in \mathcal{L}_\Box \mid \text{for any } (\cdot)^\star, \text{ we have } \mathsf{PA} \vdash (\varphi)^\star\}$$

Solovay for quantified modal logic?

Let $\mathcal{L}_{\Box,\forall}$ be the language of relational quantified modal logic:

$$\top, \text{ relation symbols, boolean connectives, } \forall x, \text{ and } \Box$$

Define arithmetical realizations $(\cdot)^\bullet$ for $\mathcal{L}_{\Box,\forall}$:

$$\begin{aligned}
\text{formulas in } \mathcal{L}_{\Box,\forall} &\to \text{formulas in } \mathcal{L}_{\mathsf{PA}}\\
n\text{-ary relation symbols} &\to \text{arithmetical formulas with } n \text{ free variables}\\
\text{boolean connectives} &\to \text{boolean connectives}\\
\forall x &\to \forall x\\
\Box &\to \mathsf{Prov}_{\mathsf{PA}}
\end{aligned}$$

Theorem (Vardanyan, 1986 and McGee, 1985)

$$\{\textit{closed } \varphi \in \mathcal{L}_{\Box,\forall} \mid \textit{for any } (\cdot)^\bullet, \; \textit{we have } \mathsf{PA} \vdash (\varphi)^\bullet\}$$

is Π^0_2-complete. Thus it is not recursively axiomatizable.

Planning an escape

Restrict $\mathcal{L}_{\Box,\forall}$ to the strictly positive fragment $\mathcal{L}_{\Diamond,\forall}$:

$$\text{Terms} ::= \text{Variables} \mid \text{Constants}$$

$$\mathcal{L}_{\Diamond,\forall} ::= \top \mid \text{relation symbols applied to Terms} \mid \varphi \wedge \varphi \mid \forall x\, \varphi \mid \Diamond\varphi$$

Define a calculus QRC$_1$ with statements $\varphi \vdash \psi$ where:

$$\varphi, \psi \in \mathcal{L}_{\Diamond,\forall}$$

The arithmetical realizations $(\cdot)^*$ for $\mathcal{L}_{\Diamond,\forall}$ send:

$$\text{formulas in } \mathcal{L}_{\Diamond,\forall} \rightarrow \text{axiomatizations of theories in } \mathcal{L}_{\mathsf{PA}}$$

Prove arithmetical soundness and completeness for QRC$_1$:

$$\mathsf{QRC}_1 = \{\varphi \vdash \psi \mid \text{for any } (\cdot)^*, \text{ we have } \mathsf{PA} \vdash (\varphi \vdash \psi)^*\}$$

QRC$_1$: Axioms and rules

$$\varphi \vdash \top \qquad \varphi \wedge \psi \vdash \varphi$$

$$\varphi \vdash \varphi \qquad \varphi \wedge \psi \vdash \psi$$

$$\frac{\varphi \vdash \psi \quad \psi \vdash \chi}{\varphi \vdash \chi} \qquad \frac{\varphi \vdash \psi \quad \varphi \vdash \chi}{\varphi \vdash \psi \wedge \chi}$$

$$\Diamond\Diamond\varphi \vdash \Diamond\varphi \qquad \frac{\varphi \vdash \psi}{\Diamond\varphi \vdash \Diamond\psi}$$

$$\frac{\varphi \vdash \psi}{\varphi \vdash \forall x\, \psi}$$

$x \notin \mathsf{fv}\, \varphi$

$$\frac{\varphi[x{\leftarrow}t] \vdash \psi}{\forall x\, \varphi \vdash \psi}$$

t free for x in φ

$$\frac{\varphi \vdash \psi}{\varphi[x{\leftarrow}t] \vdash \psi[x{\leftarrow}t]}$$

t free for x in φ and ψ

$$\frac{\varphi[x{\leftarrow}c] \vdash \psi[x{\leftarrow}c]}{\varphi \vdash \psi}$$

c not in φ nor ψ

Some provable and unprovable statements

$$\Diamond \forall x\, \varphi \vdash \forall x\, \Diamond \varphi$$

$$\forall x\, \Diamond \varphi \nvdash \Diamond \forall x\, \varphi$$

$$\frac{\varphi \vdash \psi[x \leftarrow c]}{\varphi \vdash \forall x\, \psi}$$

x not free in φ and c not in φ nor ψ

Arithmetical semantics

The arithmetical realizations $(\cdot)^*$ for $\mathcal{L}_{\Diamond,\forall}$:

$$
\begin{aligned}
\text{formulas in } \mathcal{L}_{\Diamond,\forall} &\to \text{axiomatizations of theories in } \mathcal{L}_{\mathsf{PA}} \\
\text{variables } x_i &\to \text{variables } y_i \\
\text{constants } c_i &\to \text{variables } z_i \\
(\top)^* &:= \tau_{\mathsf{PA}}(u) \\
(S(x,c))^* &:= \sigma(y,z,u) \vee \tau_{\mathsf{PA}}(u) \\
(\psi(x,c) \wedge \delta(x,c))^* &:= (\psi(x,c))^* \vee (\delta(x,c))^* \\
(\Diamond\psi(x,c))^* &:= \tau_{\mathsf{PA}}(u) \vee (u = \ulcorner \mathsf{Con}_{(\psi(x,c))^*} \top \urcorner) \\
(\forall x_i\, \psi(x,c))^* &:= \exists y_i\, (\psi(x,c))^* \\
(\varphi(x,c) \vdash \psi(x,c))^* &:= \forall \theta, y, z\, (\Box_{\psi^*(y,z)}\theta \to \Box_{\varphi^*(y,z)}\theta)
\end{aligned}
$$

Arithmetical soundness

Theorem (Arithmetical soundness)

$$\mathrm{QRC}_1 \subseteq \{\varphi \vdash \psi \mid \textit{for any } (\cdot)^*, \textit{ we have } \mathrm{PA} \vdash \forall \theta, y, z\, (\Box_{\psi^*(y,z)}\theta \to \Box_{\varphi^*(y,z)}\theta)\}$$

By induction on the QRC$_1$-proof. Here is the case of $\Diamond\Diamond\varphi \vdash \Diamond\varphi$:

- Pick any $(\cdot)^*$, reason in T, and let θ, y, z be arbitrary
- Assume $\Box_{(\Diamond\varphi)^*}\theta$
- Then $\Box_{\mathrm{PA}}(\mathrm{Con}_{\varphi^*}(\top) \to \theta)$
- By provable Σ_1-completeness, $\Box_{\mathrm{PA}}(\mathrm{Con}_{\mathrm{PA}}(\mathrm{Con}_{\varphi^*}(\top)) \to \mathrm{Con}_{\varphi^*}(\top))$
- Then $\Box_{\mathrm{PA}}(\mathrm{Con}_{\mathrm{PA}}(\mathrm{Con}_{\varphi^*}(\top)) \to \theta)$
- We conclude $\Box_{(\Diamond\Diamond\varphi)^*}\theta$

Arithmetical completeness

Theorem (Arithmetical completeness)

$$\mathrm{QRC}_1 \supseteq \{\varphi \vdash \psi \mid \textit{for any } (\cdot)^*, \ \textit{we have } T \vdash (\varphi \vdash \psi)^*\}$$

Where T is a r.e. theory extending $\mathrm{I}\Sigma_1$.

Adapt Solovay's completeness proof:

- Need Kripke completeness for QRC$_1$
- Counter models should be finite, transitive, irreflexive, rooted, and have constant domain
- Embed such models in arithmetic using the Solovay sentences λ_i
- ...

Relational models

Kripke models where:

- each world w is a first-order model with a finite domain D
- the domain D is the same for every world (new!)
- each constant symbol c and relational symbol S has a denotation at each world
- there is a transitive relation R between worlds
- constants have the same denotation at every world
- the denotation of a relation symbol depends on the world
- we use assignments g : Variables $\to D$ to interpret variables
- we abuse notation and define $g(c) :=$ denotation(c) for all assignments g and constants c

Satisfaction

Let g be a w-assignment.

$$\mathcal{M}, w \Vdash^g S(t, u) \iff \langle g(t), g(u)\rangle \in \text{denotation}_w(S)$$

$\mathcal{M}, w \Vdash^g \Diamond\varphi \iff$

there is a world v such that wRv and $\mathcal{M}, v \Vdash^g \varphi$

$\mathcal{M}, w \Vdash^g \forall x\, \varphi \iff$

for all assignments $h \sim_x g$, we have $\mathcal{M}, w \Vdash^h \varphi$

Relational soundness and completeness

Theorem (Relational soundness)

If $\varphi \vdash \psi$, then for any model $\mathcal{M}$, world w, and assignment g:

$$\mathcal{M}, w \Vdash^g \varphi \implies \mathcal{M}, w \Vdash^g \psi.$$

Theorem (Relational completeness)

If $\varphi \nvdash \psi$, then there is a finite model $\mathcal{M}$, a world w, and an assignment g such that:

$$\mathcal{M}, w \Vdash^g \varphi \quad \textit{and} \quad \mathcal{M}, w \nVdash^g \psi.$$

Since QRC$_1$ has the finite model property, it is decidable.

Proving relational completeness

- Given $\varphi \nvdash \psi$, build a counter-model
- The standard is to use term models: each world is the set of formulas true at that world
- We also want to know which formulas are *not* true at given worlds
- Our worlds are pairs of "positive" (true) and "negative" (false) formulas:

$$w = \langle w^+, w^- \rangle \qquad \text{e.g. } \langle \{\varphi\}, \{\psi\} \rangle$$

- Worlds should be *well-formed* pairs though...

Well-formed pairs

Let Λ be a set of formulas and p be a pair.

- $\Gamma \vdash \delta$ is shorthand for $(\bigwedge_{\gamma \in \Gamma} \gamma) \vdash \delta$
- p is *closed* if every formula in p is closed
- p is *consistent* if for every $\delta \in p^-$ we have $p^+ \nvdash \delta$
- p is Λ-*maximal* if for every $\varphi \in \Lambda$, either $\varphi \in p^+$ or $\varphi \in p^-$
- p is *fully witnessed* if for every formula $\forall x\, \varphi \in p^-$ there is a constant c such that $\varphi[x \leftarrow c] \in p^-$
- p is Λ-*well-formed* if it is closed, Λ-maximal, consistent and fully witnessed

Building a world from an incomplete pair

- Let Λ be a finite set of closed formulas
- Let C be a finite set of constants containing the constants in Λ and some new constants
- Let Λ_C be the closure under (closed) subformulas of Λ, and such that if $\forall x\, \varphi \in \Lambda_C$, then for every $c \in C$ we have $\varphi[x \leftarrow c] \in \Lambda_C$
- Let $p = \langle p^+, p^- \rangle$ be a closed consistent pair such that $p^+ \cup p^- \subseteq \Lambda_C$
- Goal: obtain a Λ_C-well-formed pair w extending p

Method

- Some formulas in Λ_C are consequences of p^+, and thus must be added to w^+ to preserve consistency
- We put all the other formulas of Λ_C in p^-

This Method works!

Lemma

If $|C| > 2(\text{max. constant count in } \Lambda) + 2(\text{max. } \forall\text{-depth of } \Lambda)$ and p^+ is a singleton, the Method produces a Λ_C-well-formed pair w.

- w is consistent because $\varphi \in w^+$ if and only if $p^+ \vdash \varphi$
- w is fully-witnessed because...

$$\forall x\, \varphi \in w^-$$

$$\Downarrow$$

there is some $c \in C$ s.t. c doesn't appear in $\forall x\, \varphi$ nor p^+

$$\Downarrow$$

$$p^+ \nvdash \varphi[x \leftarrow c]$$

$$\Downarrow$$

$$\varphi[x \leftarrow c] \in w^-$$

Building a counter-model

- Start with $\varphi \nvdash \psi$ (both closed)
- Build a (well-formed!) world w by extending $p := \langle \{\varphi\}, \{\psi\} \rangle$ (with $\Lambda := \{\varphi, \psi\}$ and C large enough for Λ)
- Let the domain be the set of constants C
- Let the denotation of relation symbols at w correspond to their membership in w^+
- If $\Diamond\chi \in w^+$, create a new world v_χ seen from w by Λ_C-completing

$$\langle \{\chi\}, \{\delta, \Diamond\delta \mid \Diamond\delta \in w^-\} \cup \{\Diamond\chi\} \rangle$$

- Define the domain and the denotation at v_χ like with w
- Repeat until all $\Diamond$-formulas are witnessed

Putting it together

Lemma (Truth lemma)

Let $\mathcal{M}$ be the counter-model we just built. Then for any world w, assignment g, and formula $\chi^g \in \Lambda_C$:

$$\mathcal{M}, w \Vdash^g \chi \iff \chi^g \in w^+,$$

where χ^g is χ with every free variable x replaced by $g(x)$.

Theorem (Relational completeness)

If $\varphi \nvdash \psi$, then there is a finite model $\mathcal{M}$, a world w, and an assignment g such that:

$$\mathcal{M}, w \Vdash^g \varphi \quad \text{and} \quad \mathcal{M}, w \nVdash^g \psi.$$

Arithmetical completeness proof

Theorem (Arithmetical completeness)

$$\mathsf{QRC}_1 \supseteq \{\varphi \vdash \psi \mid \textit{for any } (\cdot)^*, \ \textit{we have } T \vdash (\varphi \vdash \psi)^*\}$$

- Assume $\varphi \nvdash \psi$
- Take a (finite, transitive, irreflexive, rooted, constant domain) Kripke model $\mathcal{M}$ satisfying φ and not ψ at world 1 (the root)
- Embed $\mathcal{M}$ (with an extra world 0 pointing to the root) into the language of arithmetic, obtaining a formula λ_i representing each world i
- Define $S^\bullet$ as:

$$(S(x_k))^\bullet := \bigvee_{i \in \mathcal{M}} \left(\lambda_i \wedge \bigvee_{\langle a \rangle \in S^{\mathcal{M}_i}} \ulcorner a \urcorner = y_k \text{ mod } m \right)$$

- Prove a Truth Lemma stating (for $i > 0$) that if $i \Vdash^g \chi$ then $T \vdash \lambda_i \to \chi^\bullet[y \leftarrow \ulcorner g(x) \urcorner]$; if $i \nVdash^g \chi$ then $T \vdash \lambda_i \to \neg \chi^\bullet[y \leftarrow \ulcorner g(x) \urcorner]$

Arithmetical completeness proof (cont'ed)

Theorem (Arithmetical completeness)

$$\mathrm{QRC}_1 \supseteq \{\varphi \vdash \psi \mid \textit{for any } (\cdot)^*, \textit{ we have } T \vdash (\varphi \vdash \psi)^*\}$$

- ...
- Prove a Truth Lemma stating (for $i > 0$) that if $i \Vdash^g \chi$ then $T \vdash \lambda_i \to \chi^\bullet[y \leftarrow \ulcorner g(x) \urcorner]$; if $i \nVdash^g \chi$ then $T \vdash \lambda_i \to \neg\chi^\bullet[y \leftarrow \ulcorner g(x) \urcorner]$
- Then $T \vdash \lambda_1 \to \varphi^\bullet[y \leftarrow \ulcorner g(x) \urcorner]$ and $T \vdash \lambda_1 \to \neg\psi^\bullet[y \leftarrow \ulcorner g(x) \urcorner]$
- Prove $\mathbb{N} \models \lambda_0$
- Prove $T \vdash \lambda_0 \to \Diamond_T \lambda_1$.
- Then $T \vdash \lambda_0 \to \Diamond_T \neg(\varphi^\bullet \to \psi^\bullet)[y \leftarrow \ulcorner g(x) \urcorner]$
- Then $\mathbb{N} \models \neg\Box_T(\varphi^\bullet \to \psi^\bullet)[y \leftarrow \ulcorner g(x) \urcorner]$
- Then $T \nvdash (\varphi^\bullet \to \psi^\bullet)[y \leftarrow \ulcorner g(x) \urcorner]$

Arithmetical completeness proof (cont'ed)

Theorem (Arithmetical completeness)

$$\mathrm{QRC}_1 \supseteq \{\varphi \vdash \psi \mid \textit{for any } (\cdot)^*,\ \textit{we have } T \vdash (\varphi \vdash \psi)^*\}$$

- ...
- We have $T \nvdash (\varphi^\bullet \to \psi^\bullet)[y \leftarrow \ulcorner g(x) \urcorner]$
- Recall $(\varphi \vdash \psi)^* = \forall\, \theta, y\, (\Box_{\psi^*}\theta \to \Box_{\varphi^*}\theta)$
- Prove $T \vdash \forall\, \theta, y\, (\Box_{\varphi^*}\theta \leftrightarrow \Box_T(\varphi^\bullet \to \theta))$
- Assume towards contradiction that $T \vdash (\varphi \vdash \psi)^*$
- Then $T \vdash \forall\, \theta, y\, (\Box_T(\psi^\bullet \to \theta) \to \Box_T(\varphi^\bullet \to \theta))$
- Then $T \vdash \Box_T(\varphi^\bullet \to \psi^\bullet)[y \leftarrow \ulcorner g(x) \urcorner]$
- Then $T \vdash (\varphi^\bullet \to \psi^\bullet)[y \leftarrow \ulcorner g(x) \urcorner]$ by soundness of T
- Contradiction!

Heyting Arithmetic

Theorem

$$\mathsf{QRC}_1 = \{\varphi \vdash \psi \mid \textit{for any } (\cdot)^*, \ \textit{we have } \mathsf{PA} \vdash (\varphi \vdash \psi)^*\}$$

- $(\varphi \vdash \psi)^* = \forall\, \theta, y, z\, (\Box_{\psi^*(y,z)}\theta \to \Box_{\varphi^*(y,z)}\theta)$
- $(\varphi \vdash \psi)^*$ is Π^0_2
- PA is Π^0_2 conservative over HA

Corollary

$$\mathsf{QRC}_1 = \{\varphi \vdash \psi \mid \textit{for any } (\cdot)^*, \ \textit{we have } \mathsf{HA} \vdash (\varphi \vdash \psi)^*\}$$

- Also works with RC$_1$

In summary

- There is no quantified provability logic with $\mathcal{L}_{\Box,\forall}$

QRC$_1$:

- quantified, strictly positive provability logic with $\mathcal{L}_{\Diamond,\forall}$
- decidable
- sound and complete w.r.t. relational semantics (with constant domain models!)
- sound and complete w.r.t arithmetical semantics
- the quantified provability logic of all r.e. theories extending $\mathrm{I}\Sigma_1$
- the quantified provability logic of HA

Thank you

ana de almeida gabriel @ ub . edu

Further Reading

G. Boolos (1995)
The Logic of Provability
Cambridge University Press

A.A.B. and J.J. Joosten (2020)
Quantified Reflection Calculus with one modality
Advances in Modal Logic 13

V.A. Vardanyan (1986)
Arithmetic complexity of predicate logics of provability and their fragments
Doklady Akad. Nauk SSSR 288(1), 11–14 (Russian)
Soviet Mathematics Doklady 33, 569–572 (English)