Gödel's ('dialectica') and monotone functional interpretations of arithmetic

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February 21, 2016

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Introduction

We present two functional interpretations of arithmetic: Gödel's functional or 'dialectica' interpretation, with a full proof of its soundness in section 2 and the monotone functional interpretation, with an overview of its soundness proof in section 3. On the way to those goals, we describe weakly extensional Heyting arithmetic in all finite types, and mention some useful results in section 1. The whole work is heavily based on chapters 3, 6, 8 and 9 of [Kohlenbach, 2008]. For more details and other references, see the ones mentioned in that book.

1 Preliminaries

In this section we give some important definitions and basic results, thoroughly used in the remaining sections of this work. We start by describing the language of intuitionistic and classical logic with and without equality in 1.1. Then we move on to describe Heyting and Peano arithmetic in 1.2. Finally, we describe how to add all the finite types to our description of arithmetic, and briefly discuss how to deal with equality in higher types in 1.3.

1.1 Intuitionistic and classical logic

We start by defining intuitionistic first order logic without equality, $\mathrm{IL}_{-=}$.

Language of $\mathbf{IL}_{-=}$ ($\mathcal{L}(\mathbf{IL}_{-=})$)

- Logical symbols: $\wedge$ (conjunction), $\vee$ (disjunction), $\to$ (implication), $\perp$ (*falsum*), $\forall$ (universal quantification) and $\exists$ (existential quantification).
- Variables: $x, y, z, \ldots$
- Function symbols: for every arity $n \geq 0$ there is a countable (possibly empty) set of function symbols $F_n = \{f_1^{(n)}, f_2^{(n)}, \ldots\}$. The symbols in F_0 are called constant symbols.
- Predicate symbols: for every arity $n > 0$ there is a countable (possibly empty) set of predicate symbols $P_n = \{p_1^{(n)}, p_2^{(n)}, \ldots\}$.

Terms are defined as follows:

- Variables are terms;
- Constants are terms;
- If $t_1, \ldots, t_n$ are terms and $f \in F_n$ is an n-ary function symbol, then $f(t_1, \ldots, t_n)$ is a term.

Formulas and prime formulas as defined as follows:

- $\perp$ is a prime formula;
- If $t_1, \ldots, t_n$ are terms and $p \in P_n$ is an n-ary predicate symbol, then $p(t_1, \ldots, t_n)$ is a prime formula;
- If A, B are formulas, then $(A \wedge B)$, $(A \vee B)$ and $(A \to B)$ are formulas;
- If A is a formula and x is a variable, then $(\forall\, x\, A)$ and $(\exists\, x\, A)$ are formulas.

We further use $(\neg A)$ (negation) as shorthand for $(A \to\perp)$ and $(A \leftrightarrow B)$ (equivalence) as shorthand for $((A \to B) \wedge (B \to A))$.

We often wish to omit the parenthesis around formulas. Hence, by convention, the priority of the logical symbols is, from higher (left) to lower (right):

- $\neg$, $\forall$, $\exists$
- $\wedge$, $\vee$
- $\to$, $\leftrightarrow$

Furthermore, $\to$ associates to the right, *i.e.*, $A \to B \to C$ is to be interpreted as $A \to (B \to C)$. Using this conventions greatly saves on the number of parenthesis necessary to understand a formula.

Formulas without both $\forall$ and $\exists$ are said to be quantifier-free, and are many times marked as so with an underscore "0", *i.e.*, $A_0, B_0, \ldots$ represent quantifier-free formulas.

Definition 1.1 (Variables of a term ($\mathrm{var}(t)$), free variables of a formula ($\mathrm{fv}(A)$) and bound variables)**.** The variables of a term, $\mathrm{var}(\cdot)$, are defined as:

- $\mathrm{var}(x) = \{x\}$;
- $\mathrm{var}(c) = \emptyset$;
- $\mathrm{var}(f(t_1, \ldots, t_n)) = \bigcup_{i=1}^{n} \mathrm{var}(t_i)$.

The free variables of a formula, $\mathrm{fv}(\cdot)$, are defined as:

- $\mathrm{fv}(\perp) = \emptyset$;
- $\mathrm{fv}(p(t_1, \ldots, t_n)) = \bigcup_{i=1}^{n} \mathrm{var}(t_i)$;
- $\mathrm{fv}(A \,\Box\, B) := \mathrm{fv}(A) \cup \mathrm{fv}(B)$, $\Box \in \{\wedge, \vee, \to\}$;
- $\mathrm{fv}(\triangle\, x\, A) := \mathrm{fv}(A) \setminus \{x\}$, $\triangle \in \{\forall, \exists\}$.

Variables that are not free but do appear in the formula are said to be bound.

If a term has no variables, we say that it is a closed term. Similarly, if a formula has no free variables, we say that it is a closed formula, or a sentence. When we write $A(x)$, we mean that $x \in \mathrm{fv}(A)$, but this does not necessarily implies that there cannot be other free variables in A.

Axioms of $\mathrm{IL}_{-=}$

Contraction:	$A \to A \wedge A$	$A \vee A \to A$	
Weakening:	$A \wedge B \to A$	$A \to A \vee B$	
Symmetry:	$A \wedge B \to B \wedge A$	$A \vee B \to B \vee A$	
Ex falso quodlibet:	$\perp \to A$		
Quantifier:	$\forall x\, A \to A[t/x]$	$A[t/x] \to \exists x\, A$	Where t is free for x in A.

The notation $A[t/x]$ represents formula A where variable x is replaced in every place where it appears free by term t. The substitution can only be made when it doesn't lead to previously free variables in t becoming bound in $A[t/x]$, which we denote by t being free for x in A.

When we use the axiom $\forall x\, A \to A[t/x]$, we say that we are instantiating x by t.

Rules of $\mathrm{IL}_{-=}$

Modus ponens:

$$\frac{A, A \to B}{B}$$

Syllogism:

$$\frac{A \to B, B \to C}{A \to C}$$

Exportation:

$$\frac{A \wedge B \to C}{A \to B \to C}$$

Importation:

$$\frac{A \to B \to C}{A \wedge B \to C}$$

Expansion:

$$\frac{A \to B}{C \vee A \to C \vee B}$$

Quantifier rules:

$$\frac{B \to A}{B \to \forall x\, A}, x \notin \mathrm{fv}(B) \qquad \frac{A \to B}{\exists x\, A \to B}, x \notin \mathrm{fv}(B)$$

Remark 1.2. From the quantifier rule for $\forall$, it is easy to prove another rule:

$$\frac{A}{\forall x\, A}$$

which we use often and call abstraction, or generalization.

There are many other descriptions of intuitionistic logic without equality, using other axioms and rules. This one is particularly useful for proving theorems about it, which is our purpose. However, a natural deduction description makes proving assertions inside the language much more straightforward. See, for example, chapters 2 and 9 of [Sørensen and Urzyczyn, 2006] for a detailed overview, including descriptions of semantics over intuitionistic logic.

$\mathrm{PL}_{-=}$

Classical first order logic without equality, $\mathrm{PL}_{-=}$ is obtained from $\mathrm{IL}_{-=}$ by adding the excluded middle axiom schema:

$$A \vee \neg A$$

for every formula A.

There are other (much simpler and more common) ways of defining $\mathrm{PL}_{-=}$, taking for example $\to$, $\neg$ and $\forall$ as logical symbols and writing the others as abbreviations, as is done in part II of [Sernadas and Sernadas, 2012]. This also means that less axioms and rules are needed. However, as we will mainly focus on $\mathrm{IL}_{-=}$ here, we do not worry ourselves with them.

IL and PL

The versions of $\mathrm{IL}_{-=}$ and $\mathrm{PL}_{-=}$ with equality, IL and PL respectively, are obtained by adding a binary predicate symbol $=$ and the equality axioms:

Reflexivity: $x = x$;

Symmetry: $x = y \to y = x$;

Transitivity: $x = y \wedge y = z \to x = z$;

Substitution in functions: $\bigwedge_{i=1}^{n} x_i = y_i \to f(x_1, \ldots, x_n) = f(y_1, \ldots, y_n)$;

Substitution in predicates: $\bigwedge_{i=1}^{n} x_i = y_i \to p(x_1, \ldots, x_n) \leftrightarrow p(y_1, \ldots, y_n)$.

We often use $x \neq y$ as abbreviation for $\neg(x = y)$.

1.2 Heyting and Peano Arithmetic

We now want to be able to talk about the natural numbers and primitive recursive functions from the natural numbers to themselves.

HA

Heyting arithmetic, also known as intuitionistic arithmetic, is denoted by HA. It has the logical symbols of $\mathcal{L}(\text{IL})$, as well as the axioms and rules of IL. Besides that, it has the following:

- Function symbols:
  - 0 (zero - a constant);
  - S (successor - an unary function symbol);
  - Symbols for all the descriptions of primitive recursive functions.
- Successor axioms:
  - $S(x) \neq 0$;
  - $S(x) = S(y) \to x = y$.
- Defining equations for the primitive recursive functions;
- Induction schema:

$$A(0) \wedge \forall x\, (A(x) \to A(S(x))) \to \forall x\, A(x)$$

Remark 1.3. We could have equivalently formulated HA using the rule of induction instead of the induction schema:

$$\frac{A(0), A(x) \to A(S(x))}{A(x)}$$

Lemma 1.4. $\text{HA} \vdash \forall x\, (x = 0 \vee x \neq 0)$.

Proof. The proof is by induction on x.

If x is 0, then $(0 = 0 \vee 0 \neq 0)$ is a direct weakening of an instance of reflexivity.

We have $S(x) \neq 0$ by axiom, so with a weakening we immediately get $(S(x) = 0 \vee S(x) \neq 0)$, and consequently $(x = 0 \vee x \neq 0) \to (S(x) = 0 \vee S(x) \neq 0)$.

The result follows by the induction rule and abstraction on x. □

Definition 1.5. We define some useful primitive recursive functions, using informal recursion:

$x + y$

- $x + 0 := x$
- $x + S(y) := S(x + y)$

$x \cdot y$

- $x \cdot 0 := 0$
- $x \cdot S(y) := (x \cdot y) + x$

$\overline{\text{sign}}$

- $\overline{\text{sign}}(0) := S(0)$
- $\overline{\text{sign}}(S(x)) := 0$

$\text{pred}(x)$

- $\text{pred}(0) := 0$
- $\text{pred}(S(x)) := x$

$x \dot{-} y$

- $x \dot{-} 0 := x$
- $x \dot{-} S(y) := \text{pred}(x \dot{-} y)$

$|x - y|$

- $|x - y| := (x \dot{-} y) + (y \dot{-} x)$

Lemma 1.6.

1. $\mathrm{HA} \vdash x = y \leftrightarrow |x - y| = 0$
2. $\mathrm{HA} \vdash x = 0 \wedge y = 0 \leftrightarrow x + y = 0$
3. $\mathrm{HA} \vdash x = 0 \vee y = 0 \leftrightarrow x \cdot y = 0$
4. $\mathrm{HA} \vdash (x = 0 \to y = 0) \leftrightarrow \overline{\mathrm{sign}}(x) \cdot y = 0$

Proof. We omit the proof, which is tedious. It goes by double induction on x, y and uses Lemma 1.4. □

From now on we will often use the notation $\boldsymbol{t}$ to mean a (possibly empty) tuple of terms $t_1, \ldots, t_k$, and $|\boldsymbol{t}| := k$. In particular, $\boldsymbol{t}, \boldsymbol{s} := t_1, \ldots, t_k, s_1, \ldots, s_l$. Furthermore, we use $\triangle \boldsymbol{x}$ in the place of $\triangle x_1 \triangle x_2 \cdots \triangle x_n$, where $\triangle \in \{\forall, \exists\}$.

Proposition 1.7. Let $A_0(\boldsymbol{x})$ be a quantifier-free formula of $\mathcal{L}(\mathrm{HA})$, with all of its free variables in $\boldsymbol{x}$. Then there is a primitive recursive function represented by a symbol f in HA such that:

$$\mathrm{HA} \vdash \forall \boldsymbol{x} \, (f(\boldsymbol{x}) = 0 \leftrightarrow A_0(\boldsymbol{x}))$$

Proof. The proof is by induction on the logical structure of A_0.

Notice that, as the only propositional symbol in $\mathcal{L}(HA)$ is $=$, the prime formulas of HA are either $\perp$ or of the form $s = t$ for terms s and t. And clearly

$$\mathrm{HA} \vdash 0 = S(0) \leftrightarrow \perp$$

so even $\perp$ can be seen as an equality between terms.

The result then follows from Lemma 1.6: item 1 takes care of the base of induction, and the other three items of the steps for each logical symbol: $\wedge$, $\vee$ and $\to$, respectively. □

Corollary 1.8. If $A_0 \in \mathcal{L}(\mathrm{HA})$ is a quantifier-free formula, then:

$$\mathrm{HA} \vdash A_0 \vee \neg A_0$$

Proof. Direct from Proposition 1.7 and Lemma 1.4. □

PA

Peano (or classical) arithmetic (PA) results from HA by adding the law of excluded middle as axiom schema:

$$A \vee \neg A$$

for all formulas A.

1.3 Weakly extensional Heyting and Peano arithmetic in all finite types

Definition 1.9 (Finite types)**.** The finite types are described inductively as:

- 0 is a finite type;
- If ρ, τ are finite types, then $(\rho \to \tau)$ is a finite type.

The type 0 should be though of as the natural numbers, and the type $\rho \to \tau$ as the type of the functions from objects of type ρ to objects of type τ. The parenthesis associate to the right, and we omit them when possible, to simplify the notation.

Remark 1.10. Any type $\rho \neq 0$ can be uniquely written as $\rho = \rho_1 \to \rho_2 \to \cdots \to \rho_k \to 0$.

WE-HA$^\omega$

We now enrich $\mathrm{IL}_{-=}$ with variables $x^\rho, y^\rho, z^\rho, \dots$ and quantifiers $\forall x^\rho, \exists x^\rho$ for all types and obtain $\mathrm{IL}^\omega_{-=}$. The language $\mathcal{L}(\text{WE-HA}^\omega)$ of weakly extensional Heyting arithmetic in all finite types, WE-HA$^\omega$, is built on top of $\mathrm{IL}^\omega_{-=}$ and besides everything in $\mathrm{IL}^\omega_{-=}$, it also contains:

- Constant symbols:
  - 0^0 (zero);
  - $S^{0\to 0}$ (successor);
  - $\Pi_{\rho,\tau}$ of type $\rho \to \tau \to \rho$, for all types ρ, τ (projectors);
  - $\Sigma_{\delta,\rho,\tau}$ of type $(\delta \to \rho \to \tau) \to (\delta \to \rho) \to \delta \to \tau$, for all types δ, ρ, τ;
  - $(\boldsymbol{R_\rho}) = (R_1)_{\boldsymbol{\rho}}, \dots, (R_k)_{\boldsymbol{\rho}}$ where $\boldsymbol{\rho} = \rho_1, \dots, \rho_k$ and each R_i has type:

    $$0 \to \rho_1 \to \cdots \to \rho_k \to (\rho_1 \to \cdots \to \rho_k \to 0 \to \rho_1) \to \cdots \to (\rho_1 \to \cdots \to \rho_k \to 0 \to \rho_k) \to \rho_i$$

    (simultaneous recursors).

  Notice that we do not have any function symbols with non-zero arity. We do have typed terms though, and the type of a term determines exactly what "arguments" that term (which might be seen as a function) can "receive", or rather, be applied to. This will allow us to do everything already possible in HA (see ahead) and more.

- Predicate symbol: $=_0$ (equality of type 0).

Terms all have a type, are defined as follows:

- Variables of type ρ are terms of type ρ;
- Constants of type ρ are terms of type ρ;
- If $T^{\rho\to\sigma}$ and s^ρ are terms, then (Ts) is a term of type σ.

We think of (Ts) as "T applied to s", as if T were an unary function and s an argument. However, if $\sigma = \tau \to \delta$, the same term T could appear as $((Ts)u^\tau)^\delta$, and now it looks like it should be a binary function. In reality, all of the options above are valid term constructions, as long as the types are correct. The parenthesis associate to the left, which means that Tsu is the same as $((Ts)u)$.

We often omit the type superscript of a term, when it is possible to determine its type by the context.

Formulas and prime formulas as defined as follows:

- If s^0, t^0 are terms, then $s =_0 t$ is a prime formula;
- If A, B are formulas, then $(A \wedge B)$, $(A \vee B)$ and $(A \to B)$ are formulas;
- If A is a formula and x^ρ is a variable, then $(\forall x^\rho A)$ and $(\exists x^\rho A)$ are formulas.

If A_0 is a quantifier-free formula, $B \equiv \forall \boldsymbol{x} A_0$ and $C \equiv \exists \boldsymbol{x} A_0$, we say that B is a purely universal formula, and that C is a purely existential formula.

Besides the abbreviations for $\neg$ and $\leftrightarrow$ already introduced for $\mathrm{IL}_{-=}$, we add the following:

- $\bot$ is an abbreviation of $0 =_0 S0$;
- If $\rho = \rho_1 \to \cdots \to \rho_k \to 0$ is a type, and s, t are of type ρ, then

  $$(s =_\rho t) \equiv_{\text{abrv}} (\forall y_1^{\rho_1}, \dots, y_k^{\rho_k}\, sy_1 \dots y_k =_0 ty_1 \dots y_k)$$

  where $y_1, \dots, y_k$ are not free variables of either s or t.

We usually omit the subscript of equality, when the type is evident from the context.

Axioms and rules of WE-HA$^\omega$:

- All axioms and rules of $\mathrm{IL}^\omega_{-=}$;
- Axioms for $=_0$:

Reflexivity: $x =_0 x$;

Symmetry: $x =_0 y \to y =_0 x$;

Transitivity: $x =_0 y \wedge y =_0 z \to x =_0 z$.

- Quantifier-free rule of extensionality:

$$\frac{A_0 \to s =_\rho t}{A_0 \to r[s/x] =_\tau r[t/x]}$$

where A_0 is a quantifier-free formula, x^ρ is a variable and s^ρ, t^ρ and r^τ are terms;

- Successor axioms;
- Induction schema;
- Axioms for Π and Σ:

$$\Pi_{\rho,\tau}xy =_\rho x, \text{ for } x^\rho, y^\tau$$
$$\Sigma_{\delta,\rho,\tau}xyz =_\tau xz(yz), \text{ for } x^{\delta\to\rho\to\tau}, y^{\delta\to\rho}, z^\delta$$

- Axioms for the recursors:

Let $\boldsymbol{\rho} = \rho_1, \dots, \rho_k$ be any tuple of types. Let x^0, $\boldsymbol{y} = y_1, \dots, y_k$ with each y_i of type ρ_i and $\boldsymbol{z} = z_1, \dots, z_k$ with each z_i of type $\rho_1 \to \cdots \to \rho_k \to 0 \to \rho_i$. The axioms are:

$$\begin{array}{l} (R_i)_{\boldsymbol{\rho}} 0\boldsymbol{y}\boldsymbol{z} =_{\rho_i} y_i \\ (R_i)_{\boldsymbol{\rho}}(Sx)\boldsymbol{y}\boldsymbol{z} =_{\rho_i} z_i(\boldsymbol{R}_{\boldsymbol{\rho}}x\boldsymbol{y}\boldsymbol{z})x \end{array} \quad \text{for } i \in \{1, \dots, k\}$$

Remark 1.11. We could have equivalently defined higher type equality by induction on the type:

$$\begin{array}{l} (s =_0 t) \text{ is already defined} \\ (s =_{\rho\to\tau} t) :\equiv (\forall\, y^\rho\, sy =_\tau ty) \end{array}$$

and will use both descriptions, depending on which one is more useful in the given context.

Remark 1.12. The reflexivity, symmetry and transitivity of higher-type equality are derivable in WE-HA$^\omega$, directly from the corresponding axioms for $=_0$.

Lemma 1.13. Quantifier-free extensionality suffices to prove:

$$\frac{A_0 \to s =_\rho t}{A_0 \to (B[s/x^\rho] \leftrightarrow B[t/x^\rho])}$$

for any formula B such that s and t are free for x in B.

Remark 1.14 (Weak extensionality)**.** Lemma 1.13 is a weaker result than the way we would really like for equality at higher types to behave:

$$x =_\rho y \wedge A(x) \to A(y)$$

which is not provable in all its generality in WE-HA$^\omega$. However, we cannot add full extensionality to our system and still prove the soundness of Gödel's interpretation (Theorem 2.7), which is one of our main goals.

Definition 1.15 (λ-abstraction)**.**

- $(\lambda\, x^\rho \,.\, x)^{\rho\to\rho} := \Sigma_{\rho,\sigma\to\rho,\rho}\Pi_{\rho,\sigma\to\rho}\Pi_{\rho,\sigma}$;
- $(\lambda\, x^\rho \,.\, t^\sigma)^{\rho\to\sigma} := \Pi_{\sigma,\rho}t$, if $x \notin \mathrm{var}(t)$;
- $(\lambda\, x^\rho \,.\, t^{\sigma\to\tau}u^\sigma)^{\rho\to\tau} := \Sigma_{\rho,\sigma,\sigma\to\tau}(\lambda\, x \,.\, t)(\lambda\, x \,.\, u)$, if $x \in \mathrm{var}(tu)$.

Remark 1.16. $\mathrm{var}(\lambda\, x \,.\, t) = \mathrm{var}(t) \setminus \{x\}$, as can be clearly seen by induction on the construction of the lambda terms.

Proposition 1.17 (Combinatorial completeness)**.** WE-HA$^\omega \vdash (\lambda\, x^\rho \,.\, t^\tau)s^\rho =_\tau t[s/x]$.

Proof. The proof follows by induction on the construction of the lambda terms:

$\boxed{\lambda\, x \,.\, x}$

$$\begin{aligned}(\lambda\, x^\rho \,.\, x)s^\rho &:= \Sigma\Pi\Pi s\\ &=_\rho \Pi s(\Pi s)\\ &=_\rho s\\ &=_\rho x[s/x]\end{aligned}$$

Where the second and third equalities follow from the axioms for Σ and Π, respectively.

$\boxed{\lambda\, x \,.\, t,\ x \notin \mathrm{var}(t)}$

$$\begin{aligned}(\lambda\, x^\rho \,.\, t^\tau)s^\rho &:= \Pi ts\\ &=_\tau t\\ &=_\tau t[s/x]\end{aligned}$$

Where the last equality follows from the fact that x is not a variable of t.

$\boxed{\lambda\, x \,.\, tu,\ x \in \mathrm{var}(tu)}$

$$\begin{aligned}(\lambda\, x^\rho \,.\, t^{\sigma\to\tau}u^\sigma)s^\rho &:= \Sigma(\lambda\, x \,.\, t)(\lambda\, x \,.\, u)s\\ &=_\tau (\lambda\, x \,.\, t)s((\lambda\, x \,.\, u)s)\\ &=_\tau t[s/x](u[s/x])\\ &=_\tau (tu)[s/x]\end{aligned}$$

Where the next-to-last equality follows by induction hypothesis, noticing the association of the parenthesis on the left.

□

We often write $\lambda\, x, y \,.\, t$ as shorthand for $\lambda\, x \,.\, (\lambda\, y \,.\, t)$. Furthermore, the notation $\lambda\, \boldsymbol{x} \,.\, \boldsymbol{t}$ should be interpreted as $(\lambda\, x_1, \ldots, x_k \,.\, t_1), \ldots, (\lambda\, x_1, \ldots, x_k \,.\, t_l)$.

The expression $\boldsymbol{Ts}$ should be interpreted as $(T_1 s_1 \ldots s_n), \ldots, (T_k s_1 \ldots s_n)$.

Corollary 1.18. For every term t^τ and variable x^ρ, there exists a term T of type $\rho \to \tau$ and variables $\mathrm{var}(T) = \mathrm{var}(t) \setminus \{x\}$ such that:

$$\text{WE-HA}^\omega \vdash Ts^\rho =_\tau t[s/x]$$

Proof. Taking $T := \lambda\, x \,.\, t$, this is a direct consequence of Remark 1.16 and Proposition 1.17. □

Proposition 1.19. HA is a subsystem of WE-HA$^\omega$.

Proof. This is not too hard to see. One translates each symbol of HA into the language of WE-HA$^\omega$. The symbols 0 and S are translated by themselves. For the projectors, one uses λ-abstraction. Also with the help of λ-abstraction, composition becomes application, and primitive recursion is handled by the simultaneous recursors of WE-HA$^\omega$.

For a detailed proof see [Troelstra, 1973](1.6.9). □

Proposition 1.20. Let $A_0(\boldsymbol{x})$ be a quantifier-free formula of WE-HA$^\omega$, with free variables among $\boldsymbol{x}$. Then there exists a closed term t_{A_0} such that:

$$\text{WE-HA}^\omega \vdash \forall \boldsymbol{x}\, (t_{A_0}\boldsymbol{x} =_0 0 \leftrightarrow A_0(\boldsymbol{x}))$$

Proof. By Proposition 1.19, there are terms in WE-HA$^\omega$ for the functions from Definition 1.5. Using those terms, we simply repeat the proof from Proposition 1.7. □

Corollary 1.21. For every quantifier-free formula A_0 of WE-HA$^\omega$:

$$\begin{aligned}&\text{WE-HA}^\omega \vdash A_0 \vee \neg A_0\\ &\text{WE-HA}^\omega \vdash \neg\neg A_0 \to A_0\end{aligned}$$

Proof. Follows from Proposition 1.20, Lemma 1.4 and Proposition 1.19, which ensures that we can use Lemma 1.4. ☐

Corollary 1.22 (Elimination of $\vee$)**.** For every quantifier-free formula A_0 of WE-HA$^\omega$, there exists an equivalent quantifier-free formula B_0 without $\vee$.

Proof. Simply take $B_0 :\equiv (t_{A_0}\boldsymbol{x} =_0 0)$, as given by Proposition 1.20. This is a prime formula, and clearly doesn't have any $\vee$. ☐

Proposition 1.23 (Definition by cases)**.** For every type ρ, there exists a closed term C such that:

$$\text{WE-HA}^\omega \vdash \forall\, x^0, y^\rho, z^\rho\, [(x = 0 \to Cxyz = y) \wedge (x \neq 0 \to Cxyz = z)]$$

Proof. Let

$$C := \lambda\, x^0, y^\rho, z^\rho\, .\, R_\rho xy(\lambda\, q^\rho, r^0\, .\, z)$$

Notice that C is well defined, for R_ρ has type $0 \to \rho \to (\rho \to 0 \to \rho) \to \rho$, x has type 0, y has type ρ and $\lambda\, q^\rho, r^0\, .\, z$ has type $\rho \to 0 \to \rho$. Furthermore:

$$\begin{aligned} C0yz &=_\rho R_\rho 0y(\lambda\, q^\rho, r^0\, .\, z) \\ &=_\rho y \\ C(Sx)yz &=_\rho R_\rho(Sx)y(\lambda\, q^\rho, r^0\, .\, z) \\ &=_\rho (\lambda\, q^\rho, r^0\, .\, z)(R_\rho xy(\lambda\, q^\rho, r^0\, .\, z))x \\ &=_\rho z \end{aligned}$$

It only remains to notice that, as $x \neq_0 0 \to x = S(\text{pred}\, x)$, then $x \neq_0 0 \to Cxyz =_\rho z$. ☐

2 Gödel's Functional ('dialectica') Interpretation

On this section we start by defining Gödel's translation for every formula of WE-HA$^\omega$. We then prove two main theorems: the soundness of the translation, and the characterization theorem.

Definition 2.1 (Gödel's translation)**.** Let $A \in \mathcal{L}(\text{WE-HA}^\omega)$ be a formula. Gödel's translation A^D of A is a formula of the form

$$A^D \equiv \exists\, \boldsymbol{x}\; \forall\, \boldsymbol{y}\; A_D(\boldsymbol{x}, \boldsymbol{y})$$

where the variable tuples $\boldsymbol{x}$ and $\boldsymbol{y}$ and their types are uniquely defined by the logical structure of A, and $A_D(\boldsymbol{x}, \boldsymbol{y})$ is a quantifier-free formula. Here is the definition of A^D and A_D (omitting the times):

- If A is a prime formula, $A^D :\equiv A_D :\equiv A$

Let $A^D \equiv \exists\, \boldsymbol{x}\; \forall\, \boldsymbol{y}\; A_D(\boldsymbol{x}, \boldsymbol{y})$ and $B^D \equiv \exists\, \boldsymbol{u}\; \forall\, \boldsymbol{v}\; B_D(\boldsymbol{u}, \boldsymbol{v})$.

- $\begin{aligned}[t] [A \wedge B]^D &:\equiv \exists\, \boldsymbol{x}, \boldsymbol{u}\; \forall\, \boldsymbol{y}, \boldsymbol{v}\, [A \wedge B]_D \\ &:\equiv \exists\, \boldsymbol{x}, \boldsymbol{u}\; \forall\, \boldsymbol{y}, \boldsymbol{v}\, [A_D(\boldsymbol{x}, \boldsymbol{y}) \wedge B_D(\boldsymbol{u}, \boldsymbol{v})] \end{aligned}$

- $\begin{aligned}[t] [A \vee B]^D &:\equiv \exists\, z^0, \boldsymbol{x}, \boldsymbol{u}\; \forall\, \boldsymbol{y}, \boldsymbol{v}\, [A \vee B]_D \\ &:\equiv \exists\, z^0, \boldsymbol{x}, \boldsymbol{u}\; \forall\, \boldsymbol{y}, \boldsymbol{v}\, [(z = 0 \to A_D(\boldsymbol{x}, \boldsymbol{y})) \wedge (z \neq 0 \to B_D(\boldsymbol{u}, \boldsymbol{v}))] \end{aligned}$

- $\begin{aligned}[t] [A \to B]^D &:\equiv \exists\, \boldsymbol{U}, \boldsymbol{Y}\; \forall\, \boldsymbol{x}, \boldsymbol{v}\, [A \to B]_D \\ &:\equiv \exists\, \boldsymbol{U}, \boldsymbol{Y}\; \forall\, \boldsymbol{x}, \boldsymbol{v}\, [A_D(\boldsymbol{x}, \boldsymbol{Y}\boldsymbol{x}\boldsymbol{v}) \to B_D(\boldsymbol{U}\boldsymbol{x}, \boldsymbol{v})] \end{aligned}$

- $\begin{aligned}[t] [\exists\, z^\rho\, A(z)]^D &:\equiv \exists\, z, \boldsymbol{x}\; \forall\, \boldsymbol{y}\, [\exists\, z\, A(z)]_D \\ &:\equiv \exists\, z, \boldsymbol{x}\; \forall\, \boldsymbol{y}\, A_D(\boldsymbol{x}, \boldsymbol{y}, z) \end{aligned}$

- $\begin{aligned}[t] [\forall\, z^\rho\, A(z)]^D &:\equiv \exists\, \boldsymbol{X}\; \forall\, z, \boldsymbol{y}\, [\forall\, z\, A(z)]_D \\ &:\equiv \exists\, \boldsymbol{X}\; \forall\, z, \boldsymbol{y}\, A_D(\boldsymbol{X}z, \boldsymbol{y}, z) \end{aligned}$

Notice that the tuples of variables that are quantified in A^D should not contain any of the free variables of A, and to find them one should start the translation from the inside, *i.e.*, with the prime formulas.

Remark 2.2.

1. $(A^D)^D \equiv A^D$, and consequently $(A \,\square\, B)^D \equiv (A^D \,\square\, B^D)^D$, for $\square \in \{\wedge, \vee, \rightarrow\}$.
2. If A is a quantifier-free formula without $\vee$, then $A^D \equiv A$.
3. If $A \equiv \triangle\, \boldsymbol{x}\, B_0$ where $\triangle \in \{\forall, \exists\}$ and B_0 is a quantifier-free formula without $\vee$, then $A^D \equiv A$.

Definition 2.3 (AC^ω)**.** The schema of choice, AC^ω, is the union for all finite types ρ and τ of:

$$\mathrm{AC}^{\rho,\tau} : \forall\, x^\rho\; \exists\, y^\tau\; A(x, y) \rightarrow \exists\, Y^{\rho\rightarrow\tau}\; \forall\, x^\rho\; A(x, Yx)$$

where A is any formula of WE-HA$^\omega$.

Definition 2.4 (M^ω)**.** Markov's principle, M^ω, is the union for all tuples of finite types $\boldsymbol{\rho}$ of:

$$\mathrm{M}^{\boldsymbol{\rho}} : \neg \forall\, \boldsymbol{x}^{\boldsymbol{\rho}}\, A_0(\boldsymbol{x}) \rightarrow \exists\, \boldsymbol{x}^{\boldsymbol{\rho}}\, \neg A_0(\boldsymbol{x})$$

where A_0 is a quantifier-free formula, possibly with more free variables other than $\boldsymbol{x}$.

Remark 2.5. In [Kohlenbach, 2008], Markov's principle is stated in a different form, namely:

$$\mathrm{M'}^{\boldsymbol{\rho}} : \neg\neg \exists\, \boldsymbol{x}^{\boldsymbol{\rho}}\, A_0(\boldsymbol{x}) \rightarrow \exists\, \boldsymbol{x}^{\boldsymbol{\rho}}\, A_0(\boldsymbol{x})$$

Both statements are intuitionistically equivalent (using the fact that quantifier-free formulas are stable - Corollary 1.21), and we use the one in Definition 2.4 because it makes the proofs bellow more direct.

Definition 2.6 ($\mathrm{IP}^\omega_\forall$)**.** The independence of premise schema for purely universal premises, $\mathrm{IP}^\omega_\forall$, is the union for all finite types ρ of:

$$\mathrm{IP}^\rho_\forall : (\forall\, \boldsymbol{x}\, A_0(\boldsymbol{x}) \rightarrow \exists\, y^\rho\, B(y)) \rightarrow \exists\, y^\rho\, (\forall\, \boldsymbol{x}\, A_0(\boldsymbol{x}) \rightarrow B(y))$$

where $A_0(\boldsymbol{x})$ is a quantifier-free formula and y is not free in A_0.

Theorem 2.7 (Soundness of Gödel's translation)**.** Let $\mathcal{P}$ be a set of purely universal sentences of $\mathcal{L}$(WE-HA$^\omega$) and $A(\boldsymbol{a}) \in \mathcal{L}$(WE-HA$^\omega$) containing only $\boldsymbol{a}$ free. Then:

$$\text{WE-HA}^\omega + \mathrm{AC}^\omega + \mathrm{IP}^\omega_\forall + \mathrm{M}^\omega + \mathcal{P} \vdash A(\boldsymbol{a})$$
$$\text{implies}$$
$$\text{WE-HA}^\omega + \mathcal{P} \vdash \forall\, \boldsymbol{y}\, A_D(\boldsymbol{T}\boldsymbol{a}, \boldsymbol{y}, \boldsymbol{a})$$

where $\boldsymbol{T}$ is a tuple of closed terms which can be extracted from a proof of $A(\boldsymbol{a})$.

Proof. The goal is to give a suitable tuple of closed terms $\boldsymbol{T}$ for each axiom and rule possibly used in the proof of $A(\boldsymbol{a})$. Each term in $\boldsymbol{T}$ will be interpreted as a "function" with input the free variables $\boldsymbol{a}$, which will do the part of the existentially quantified variables in A^D, such that $\forall\, \boldsymbol{y}\, A_D(\boldsymbol{T}\boldsymbol{a}, \boldsymbol{y}, \boldsymbol{a})$ is provable in WE-HA$^\omega + \mathcal{P}$.

In some steps of the proof we will need a dummy term, that doesn't need to have any particular property besides being well-defined. We use $\mathscr{O}^\rho := \lambda\, x_1^{\rho_1}, \ldots, x_k^{\rho_k}\,.\, 0^0$ for $\rho = \rho_1 \rightarrow \cdots \rightarrow \rho_k \rightarrow 0$.

We will omit the types, so as to avoid overloading the notation.

$\boxed{A \rightarrow A \wedge A}$

The first step to find $[A \rightarrow A \wedge A]^D$ is to find A^D. Each of the three instances of A^D should have different quantified variables, so that there is no confusion. So let's say that $A^D \equiv \exists\, \boldsymbol{x}\; \forall\, \boldsymbol{y}\, A_D(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a})$, and use the pairs $(\boldsymbol{u}, \boldsymbol{v})$ and $(\boldsymbol{q}, \boldsymbol{r})$ for the other two instances (effectively obtaining two α-equivalent versions of A^D). During the rest of the proof we will always use the pairs of variables $(\boldsymbol{x}, \boldsymbol{y}), (\boldsymbol{u}, \boldsymbol{v}), (\boldsymbol{q}, \boldsymbol{r}), (\boldsymbol{o}, \boldsymbol{p})$, in this order.

Then:

$$\begin{aligned}[A \wedge A]^D &\equiv [\exists\, \boldsymbol{u}\; \forall\, \boldsymbol{v}\, A_D(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{a}) \wedge \exists\, \boldsymbol{q}\; \forall\, \boldsymbol{r}\, A_D(\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{a})]^D \\ &\equiv \exists\, \boldsymbol{u}, \boldsymbol{q}\; \forall\, \boldsymbol{v}, \boldsymbol{r}\, (A_D(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{a}) \wedge A_D(\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{a}))\end{aligned}$$

and so:

$$[A \to A \wedge A]^D \equiv [\exists \boldsymbol{x}\, \forall \boldsymbol{y}\, A_D(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a}) \to \exists \boldsymbol{u}, \boldsymbol{q}\, \forall \boldsymbol{v}, \boldsymbol{r}\, (A_D(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{a}) \wedge A_D(\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{a}))]^D$$
$$\equiv \exists \boldsymbol{U}, \boldsymbol{Q}, \boldsymbol{Y}\, \forall \boldsymbol{x}, \boldsymbol{v}, \boldsymbol{r}\, (A_D(\boldsymbol{x}, \boldsymbol{Yxvr}, \boldsymbol{a}) \to A_D(\boldsymbol{Ux}, \boldsymbol{v}, \boldsymbol{a}) \wedge A_D(\boldsymbol{Qx}, \boldsymbol{r}, \boldsymbol{a})) \quad (2.1)$$

Now we need to chose closed terms $\boldsymbol{T_U}$, $\boldsymbol{T_Q}$ and $\boldsymbol{T_Y}$ such that (2.1) is provable in WE-HA$^\omega + \mathcal{P}$. Consider the following:

$$\boldsymbol{T_U} := \lambda\, \boldsymbol{a}, \boldsymbol{x} \,.\, \boldsymbol{x}$$
$$\boldsymbol{T_Q} := \lambda\, \boldsymbol{a}, \boldsymbol{x} \,.\, \boldsymbol{x}$$
$$\boldsymbol{T_Y} := \lambda\, \boldsymbol{a}, \boldsymbol{x}, \boldsymbol{v}, \boldsymbol{r} \,.\, \begin{cases} \boldsymbol{v} & \text{if } \neg A_D(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{a}) \\ \boldsymbol{r} & \text{if } A_D(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{a}) \end{cases}$$

Notice that we can define $\boldsymbol{T_Y}$ as shown, because, as $A_D(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{a})$ is a quantifier-free formula, by Proposition 1.20 we know that there exists a closed term t such that

$$\text{WE-HA}^\omega \vdash t\boldsymbol{xva} = 0 \leftrightarrow A_D(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{a})$$

and hence checking whether $A_D(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{a})$ is the same as checking if $t = 0$, which we know how to do due to Proposition 1.23.

Finally, notice that replacing the existentially quantified $\boldsymbol{U}$, $\boldsymbol{Q}$ and $\boldsymbol{Y}$ by their respective terms followed by $\boldsymbol{a}$ in (2.1) we obtain:

$$\begin{cases} \forall \boldsymbol{x}, \boldsymbol{v}, \boldsymbol{r}\, (A_D(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{a}) \to A_D(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{a}) \wedge A_D(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{a})) & \text{if } \neg A_D(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{a}) \\ \forall \boldsymbol{x}, \boldsymbol{v}, \boldsymbol{r}\, (A_D(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{a}) \to A_D(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{a}) \wedge A_D(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{a})) & \text{if } A_D(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{a}) \end{cases}$$

and in both cases the formulas are clearly provable in WE-HA$^\omega$.

$A \vee A \to A$

Notice that:

$$[A \vee A \to A]^D \equiv$$
$$\equiv [\exists z^0, \boldsymbol{x}, \boldsymbol{u}\, \forall \boldsymbol{y}, \boldsymbol{v}\, ((z = 0 \to A_D(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a})) \wedge (z \neq 0 \to A_D(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{a}))) \to \exists \boldsymbol{q}\, \forall \boldsymbol{r}\, A_D(\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{a})]^D$$
$$\equiv \exists \boldsymbol{Q}, \boldsymbol{Y}, \boldsymbol{V}\, \forall z, \boldsymbol{x}, \boldsymbol{u}, \boldsymbol{r}$$
$$((z = 0 \to A_D(\boldsymbol{x}, \boldsymbol{Y}z\boldsymbol{xur}, \boldsymbol{a})) \wedge (z \neq 0 \to A_D(\boldsymbol{u}, \boldsymbol{V}z\boldsymbol{xur}, \boldsymbol{a})) \to A_D(\boldsymbol{Q}z\boldsymbol{xu}, \boldsymbol{r}, \boldsymbol{a})) \quad (2.2)$$

Consider the following terms:

$$\boldsymbol{T_Q} := \lambda\, \boldsymbol{a}, z, \boldsymbol{x}, \boldsymbol{u} \,.\, \begin{cases} \boldsymbol{x} & \text{if } z = 0 \\ \boldsymbol{u} & \text{if } z \neq 0 \end{cases}$$
$$\boldsymbol{T_Y} := \lambda\, \boldsymbol{a}, z, \boldsymbol{x}, \boldsymbol{u}, \boldsymbol{r} \,.\, \boldsymbol{r}$$
$$\boldsymbol{T_V} := \lambda\, \boldsymbol{a}, z, \boldsymbol{x}, \boldsymbol{u}, \boldsymbol{r} \,.\, \boldsymbol{r}$$

where the definition by cases of $\boldsymbol{T_Q}$ is possible due to Proposition 1.23.

Then replacing $\boldsymbol{Q}$, $\boldsymbol{Y}$ and $\boldsymbol{V}$ by their respective terms followed by $\boldsymbol{a}$ in (2.2) we obtain:

$$\begin{cases} \forall z, \boldsymbol{x}, \boldsymbol{u}, \boldsymbol{r}\, ((z = 0 \to A_D(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{a})) \wedge (z \neq 0 \to A_D(\boldsymbol{u}, \boldsymbol{r}, \boldsymbol{a})) \to A_D(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{a})) & \text{if } z = 0 \\ \forall z, \boldsymbol{x}, \boldsymbol{u}, \boldsymbol{r}\, ((z = 0 \to A_D(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{a})) \wedge (z \neq 0 \to A_D(\boldsymbol{u}, \boldsymbol{r}, \boldsymbol{a})) \to A_D(\boldsymbol{u}, \boldsymbol{r}, \boldsymbol{a})) & \text{if } z \neq 0 \end{cases}$$

and in both cases the formulas are clearly provable in WE-HA$^\omega$.

$A \wedge B \to A$

Let $\boldsymbol{a'}$ be the free variables of A, $\boldsymbol{a''}$ be the free variables of B and $\boldsymbol{a} := \boldsymbol{a'}, \boldsymbol{a''}$.

Notice that:

$$
\begin{aligned}
[A \wedge B \to A]^D &\equiv [\exists \boldsymbol{x}, \boldsymbol{u}\, \forall \boldsymbol{y}, \boldsymbol{v}\, (A_D(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a'}) \wedge B_D(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{a''})) \to \exists \boldsymbol{q}\, \forall \boldsymbol{r}\, A_D(\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{a'})]^D \\
&\equiv \exists \boldsymbol{Q}, \boldsymbol{Y}, \boldsymbol{V}\ \forall \boldsymbol{x}, \boldsymbol{u}, \boldsymbol{r}\, (A_D(\boldsymbol{x}, \boldsymbol{Yxur}, \boldsymbol{a'}) \wedge B_D(\boldsymbol{u}, \boldsymbol{Vxur}, \boldsymbol{a''}) \to A_D(\boldsymbol{Qxu}, \boldsymbol{r}, \boldsymbol{a'}))
\end{aligned} \tag{2.3}
$$

Let:

$$
\begin{aligned}
\boldsymbol{T_Q} &:= \lambda\, \boldsymbol{a}, \boldsymbol{x}, \boldsymbol{u}\, .\, \boldsymbol{x} \\
\boldsymbol{T_Y} &:= \lambda\, \boldsymbol{a}, \boldsymbol{x}, \boldsymbol{u}, \boldsymbol{r}\, .\, \boldsymbol{r} \\
\boldsymbol{T_V} &:= \lambda\, \boldsymbol{a}, \boldsymbol{x}, \boldsymbol{u}, \boldsymbol{r}\, .\, \mathscr{O}
\end{aligned}
$$

Then, replacing each quantified variable by its term followed by $\boldsymbol{a}$, we obtain:

$$\forall \boldsymbol{x}, \boldsymbol{u}, \boldsymbol{r}\, (A_D(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{a'}) \wedge B_D(\boldsymbol{u}, \mathscr{O}, \boldsymbol{a''}) \to A_D(\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{a'}))$$

which is a generalized instance of $A \wedge B \to A$, and hence provable in WE-HA$^\omega$.

$\boxed{A \to A \vee B}$

$$
\begin{aligned}
&[A \to A \vee B]^D \equiv \\
&\quad \equiv [\exists \boldsymbol{x}\, \forall \boldsymbol{y}\, A_D(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a'})]^D \to [\exists z^0, \boldsymbol{u}, \boldsymbol{q}\, \forall \boldsymbol{v}, \boldsymbol{r}\, ((z = 0 \to A_D(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{a'})) \wedge (z \neq 0 \to B_D(\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{a''})))]^D \\
&\quad \equiv \exists Z, \boldsymbol{U}, \boldsymbol{Q}, \boldsymbol{Y}\ \forall \boldsymbol{x}, \boldsymbol{v}, \boldsymbol{r} \\
&\qquad (A_D(\boldsymbol{x}, \boldsymbol{Yxvr}, \boldsymbol{a'}) \to (Z\boldsymbol{x} = 0 \to A_D(\boldsymbol{Ux}, \boldsymbol{v}, \boldsymbol{a'})) \wedge (Z\boldsymbol{x} \neq 0 \to B_D(\boldsymbol{Qx}, \boldsymbol{v}, \boldsymbol{a''})))
\end{aligned} \tag{2.4}
$$

Let:

$$
\begin{aligned}
T_Z &:= \lambda\, \boldsymbol{a}, \boldsymbol{x}\, .\, 0^0 \\
\boldsymbol{T_U} &:= \lambda\, \boldsymbol{a}, \boldsymbol{x}\, .\, \boldsymbol{x} \\
\boldsymbol{T_Q} &:= \lambda\, \boldsymbol{a}, \boldsymbol{x}\, .\, \mathscr{O} \\
\boldsymbol{T_Y} &:= \lambda\, \boldsymbol{a}, \boldsymbol{x}, \boldsymbol{v}, \boldsymbol{r}\, .\, \boldsymbol{v}
\end{aligned}
$$

Then:

$$\forall \boldsymbol{x}, \boldsymbol{v}, \boldsymbol{r}\, (A_D(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{a'}) \to (0 = 0 \to A_D(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{a'})) \wedge (0 \neq 0 \to B_D(\mathscr{O}, \boldsymbol{v}, \boldsymbol{a''})))$$

$\boxed{A \wedge B \to B \wedge A}$

$$
\begin{aligned}
&[A \wedge B \to B \wedge A]^D \equiv \\
&\quad \equiv [\exists \boldsymbol{x}, \boldsymbol{u}\, \forall \boldsymbol{y}, \boldsymbol{v}\, A_D(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a'}) \wedge B_D(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{a''}) \to \exists \boldsymbol{q}, \boldsymbol{o}\, \forall \boldsymbol{r}, \boldsymbol{p}\, B_D(\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{a''}) \wedge A_D(\boldsymbol{o}, \boldsymbol{p}, \boldsymbol{a'})]^D \\
&\quad \equiv \exists \boldsymbol{Q}, \boldsymbol{O}, \boldsymbol{Y}, \boldsymbol{V}\ \forall \boldsymbol{x}, \boldsymbol{u}, \boldsymbol{r}, \boldsymbol{p} \\
&\qquad (A_D(\boldsymbol{x}, \boldsymbol{Yxurp}, \boldsymbol{a'}) \wedge B_D(\boldsymbol{u}, \boldsymbol{Vxurp}, \boldsymbol{a''}) \to B_D(\boldsymbol{Qxu}, \boldsymbol{r}, \boldsymbol{a''}) \wedge A_D(\boldsymbol{Oxu}, \boldsymbol{p}, \boldsymbol{a'}))
\end{aligned} \tag{2.5}
$$

Let:

$$
\begin{aligned}
\boldsymbol{T_Q} &:= \lambda\, \boldsymbol{a}, \boldsymbol{x}, \boldsymbol{u}\, .\, \boldsymbol{u} \\
\boldsymbol{T_O} &:= \lambda\, \boldsymbol{a}, \boldsymbol{x}, \boldsymbol{u}\, .\, \boldsymbol{x} \\
\boldsymbol{T_Y} &:= \lambda\, \boldsymbol{a}, \boldsymbol{x}, \boldsymbol{u}, \boldsymbol{r}, \boldsymbol{p}\, .\, \boldsymbol{p} \\
\boldsymbol{T_V} &:= \lambda\, \boldsymbol{a}, \boldsymbol{x}, \boldsymbol{u}, \boldsymbol{r}, \boldsymbol{p}\, .\, \boldsymbol{r}
\end{aligned}
$$

Then $\forall \boldsymbol{x}, \boldsymbol{u}, \boldsymbol{r}, \boldsymbol{p}\, (A_D(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{a'}) \wedge B_D(\boldsymbol{u}, \boldsymbol{r}, \boldsymbol{a''}) \to B_D(\boldsymbol{u}, \boldsymbol{r}, \boldsymbol{a''}) \wedge A_D(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{a'}))$.

$\boxed{A \vee B \to B \vee A}$

$$\begin{aligned}
&[A \vee B \to B \vee A]^D \equiv \\
&\quad \equiv [\exists\, z^0, \boldsymbol{x}, \boldsymbol{u}\ \forall\, \boldsymbol{y}, \boldsymbol{v}\, ((z = 0 \to A_D(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a'})) \wedge (z \neq 0 \to B_D(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{a''}))) \\
&\qquad\qquad \to \exists\, w^0, \boldsymbol{q}, \boldsymbol{o}\ \forall\, \boldsymbol{r}, \boldsymbol{p}\, ((w = 0 \to B_D(\boldsymbol{q}, \boldsymbol{r}, \boldsymbol{a''})) \wedge (w \neq 0 \to A_D(\boldsymbol{o}, \boldsymbol{p}, \boldsymbol{a'})))]^D \\
&\quad \equiv \exists\, W, \boldsymbol{Q}, \boldsymbol{O}, \boldsymbol{Y}, \boldsymbol{V}\ \forall\, z, \boldsymbol{x}, \boldsymbol{u}, \boldsymbol{r}, \boldsymbol{p}\, ((z = 0 \to A_D(\boldsymbol{x}, \boldsymbol{Y} z\boldsymbol{x}\boldsymbol{u}\boldsymbol{r}\boldsymbol{p}, \boldsymbol{a'})) \wedge (z \neq 0 \to B_D(\boldsymbol{u}, \boldsymbol{V} z\boldsymbol{x}\boldsymbol{u}\boldsymbol{r}\boldsymbol{p}, \boldsymbol{a''})) \\
&\qquad\qquad \to (W z\boldsymbol{x}\boldsymbol{u} = 0 \to B_D(\boldsymbol{Q} z\boldsymbol{x}\boldsymbol{u}, \boldsymbol{r}, \boldsymbol{a''})) \wedge (W z\boldsymbol{x}\boldsymbol{u} \neq 0 \to A_D(\boldsymbol{O} z\boldsymbol{x}\boldsymbol{u}, \boldsymbol{p}, \boldsymbol{a'})))
\end{aligned} \tag{2.6}$$

Let:

$$\begin{aligned}
T_W &:= \lambda\, \boldsymbol{a}, z, \boldsymbol{x}, \boldsymbol{u}\, .\, \overline{\mathrm{sign}}(z) \\
\boldsymbol{T_Q} &:= \lambda\, \boldsymbol{a}, z, \boldsymbol{x}, \boldsymbol{u}\, .\, \boldsymbol{u} \\
\boldsymbol{T_O} &:= \lambda\, \boldsymbol{a}, z, \boldsymbol{x}, \boldsymbol{u}\, .\, \boldsymbol{x} \\
\boldsymbol{T_Y} &:= \lambda\, \boldsymbol{a}, z, \boldsymbol{x}, \boldsymbol{u}, \boldsymbol{r}, \boldsymbol{p}\, .\, \boldsymbol{p} \\
\boldsymbol{T_V} &:= \lambda\, \boldsymbol{a}, z, \boldsymbol{x}, \boldsymbol{u}, \boldsymbol{r}, \boldsymbol{p}\, .\, \boldsymbol{r}
\end{aligned}$$

Then:

$$\begin{aligned}
&\forall\, z, \boldsymbol{x}, \boldsymbol{u}, \boldsymbol{r}, \boldsymbol{p}\, ((z = 0 \to A_D(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{a'})) \wedge (z \neq 0 \to B_D(\boldsymbol{u}, \boldsymbol{r}, \boldsymbol{a''})) \\
&\qquad \to (z \neq 0 \to B_D(\boldsymbol{u}, \boldsymbol{r}, \boldsymbol{a''})) \wedge (z = 0 \to A_D(\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{a'})))
\end{aligned}$$

$\boxed{\perp\to A}$

$$\begin{aligned}
[\perp\to A]^D &\equiv [\perp\to \exists\, \boldsymbol{x}\ \forall\, \boldsymbol{y}\, A_D(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a})]^D \\
&\equiv \exists\, \boldsymbol{x}\ \forall\, \boldsymbol{y}\, (\perp\to A_D(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a}))
\end{aligned} \tag{2.7}$$

Let $\boldsymbol{T_x} := \lambda\, \boldsymbol{a}\, .\, \mathscr{O}$. We obtain $\forall\, \boldsymbol{y}\, (\perp\to A_D(\mathscr{O}, \boldsymbol{y}, \boldsymbol{a}))$, clearly provable in WE-HA$^\omega$.

$\boxed{\forall\, z\, A \to A[t/z],\ t \text{ free for } z \text{ in } A}$

Let the free variables of A be in $\boldsymbol{a'}, z$, and the variables of t be $\boldsymbol{a''}$ (possibly including z). Then $\boldsymbol{a} = \boldsymbol{a'}, \boldsymbol{a''}$ are the free variables of $\forall\, z\, A \to A[t/z]$.

$$\begin{aligned}
[\forall\, z\, A \to A[t/z]]^D &\equiv [\exists\, \boldsymbol{X}\ \forall\, z, \boldsymbol{y}\, A_D(\boldsymbol{X} z, \boldsymbol{y}, \boldsymbol{a'}, \boldsymbol{z}) \to \exists\, \boldsymbol{u}\ \forall\, \boldsymbol{v}\, A_D(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{a'}, t)]^D \\
&\equiv \exists\, \boldsymbol{U}, Z, \boldsymbol{Y}\ \forall\, \boldsymbol{X}, \boldsymbol{v}\, (A_D(\boldsymbol{X}(Z\boldsymbol{X}\boldsymbol{v}), \boldsymbol{Y}\boldsymbol{X}\boldsymbol{v}, \boldsymbol{a'}, Z\boldsymbol{X}\boldsymbol{v}) \to A_D(\boldsymbol{U}\boldsymbol{X}, \boldsymbol{v}, \boldsymbol{a'}, t))
\end{aligned} \tag{2.8}$$

Let:

$$\begin{aligned}
\boldsymbol{T_U} &:= \lambda\, \boldsymbol{a}, \boldsymbol{X}\, .\, \boldsymbol{X} t \\
T_Z &:= \lambda\, \boldsymbol{a}, \boldsymbol{X}, \boldsymbol{v}\, .\, t \\
\boldsymbol{T_Y} &:= \lambda\, \boldsymbol{a}, \boldsymbol{X}, \boldsymbol{v}\, .\, \boldsymbol{v}
\end{aligned}$$

Then $\forall\, \boldsymbol{X}, \boldsymbol{v}\, (A_D(\boldsymbol{X} t, \boldsymbol{v}, \boldsymbol{a'}, t) \to A_D(\boldsymbol{X} t, \boldsymbol{v}, \boldsymbol{a'}, t))$.

$\boxed{A[t/z] \to \exists\, z\, A,\ t \text{ free for } z \text{ in } A}$

Let the free variables in A be in $\boldsymbol{a'}, z$, and the variables of t be $\boldsymbol{a''}$ (possibly including z). Then $\boldsymbol{a} = \boldsymbol{a'}, \boldsymbol{a''}$ are the free variables of $A[t/z] \to \exists\, z\, A$.

$$\begin{aligned}
[A[t/z] \to \exists\, z\, A]^D &\equiv [\exists\, \boldsymbol{x}\ \forall\, \boldsymbol{y}\, A_D(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a'}, t) \to \exists\, z, \boldsymbol{u}\ \forall\, \boldsymbol{v}\, A_D(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{a'}, z)]^D \\
&\equiv \exists\, Z, \boldsymbol{U}, \boldsymbol{Y}\ \forall\, \boldsymbol{x}, \boldsymbol{v}\, (A_D(\boldsymbol{x}, \boldsymbol{Y}\boldsymbol{x}\boldsymbol{v}, \boldsymbol{a'}, t) \to A_D(\boldsymbol{U}\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{a'}, Z\boldsymbol{x}))
\end{aligned} \tag{2.9}$$

Let:

$$T_Z := \lambda\, \boldsymbol{a}, \boldsymbol{x} \,.\, t$$
$$\boldsymbol{T_U} := \lambda\, \boldsymbol{a}, \boldsymbol{x} \,.\, \boldsymbol{x}$$
$$\boldsymbol{T_Y} := \lambda\, \boldsymbol{a}, \boldsymbol{x}, \boldsymbol{v} \,.\, \boldsymbol{v}$$

Then $\forall\, \boldsymbol{x}, \boldsymbol{v}\, (A_D(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{a'}, t) \to A_D(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{a'}, t))$.

Modus ponens

Recall the *modus ponens* rule:

$$\frac{A, A \to B}{B}$$

Notice that:

$$A^D \equiv \exists\, \boldsymbol{x}\ \forall\, \boldsymbol{y}\, A_D(\boldsymbol{x}, \boldsymbol{y}, \boldsymbol{a'})$$
$$[A \to B]^D \equiv \exists\, \boldsymbol{U}, \boldsymbol{Y}\ \forall\, \boldsymbol{x}, \boldsymbol{v}\, (A_D(\boldsymbol{x}, \boldsymbol{Yxv}, \boldsymbol{a'}) \to B_D(\boldsymbol{Ux}, \boldsymbol{v}, \boldsymbol{a''}))$$
$$B^D \equiv \exists\, \boldsymbol{u}\ \forall\, \boldsymbol{v}\, B_D(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{a''})$$

So, by induction hypothesis, there are closed terms $\boldsymbol{T}_1$, $\boldsymbol{T}_2$ and $\boldsymbol{T}_3$ such that:

$$\forall\, \boldsymbol{y}\, A_D(\boldsymbol{T}_1\boldsymbol{a'}, \boldsymbol{y}, \boldsymbol{a'}) \tag{2.10}$$
$$\forall\, \boldsymbol{x}, \boldsymbol{v}\, (A_D(\boldsymbol{x}, \boldsymbol{T}_3\boldsymbol{axv}, \boldsymbol{a'}) \to B_D(\boldsymbol{T}_2\boldsymbol{ax}, \boldsymbol{v}, \boldsymbol{a''})) \tag{2.11}$$

Let $\boldsymbol{o}$ be the result of replacing each variable that appears in $\boldsymbol{a'}$ and not in $\boldsymbol{a''}$ by $\mathscr{O}$ of the appropriate type, and leaving the others alone. Take $\boldsymbol{T}_4$ as:

$$\boldsymbol{T}_4 := \lambda\, \boldsymbol{a''} \,.\, \boldsymbol{T}_2\boldsymbol{oa''}(\boldsymbol{T}_1\boldsymbol{o})$$

Instantiate $\boldsymbol{x}$ in (2.11) by $\boldsymbol{T}_1\boldsymbol{o}$ and $\boldsymbol{v}$ by itself, obtaining:

$$A_D(\boldsymbol{T}_1\boldsymbol{o}, \boldsymbol{T}_3\boldsymbol{oa''}(\boldsymbol{T}_1\boldsymbol{o})\boldsymbol{v}, \boldsymbol{o}) \to B_D(\boldsymbol{T}_2\boldsymbol{oa''}(\boldsymbol{T}_1\boldsymbol{o}), \boldsymbol{v}, \boldsymbol{a''})$$

Now instantiate $\boldsymbol{y}$ in (2.10) by $\boldsymbol{T}_3\boldsymbol{oa''}(\boldsymbol{T}_1\boldsymbol{o})\boldsymbol{v}$, thus obtaining $A_D(\boldsymbol{T}_1\boldsymbol{o}, \boldsymbol{T}_3\boldsymbol{oa''}(\boldsymbol{T}_1\boldsymbol{o})\boldsymbol{v}, \boldsymbol{o})$. By *modus ponens*, we are able to conclude $B_D(\boldsymbol{T}_2\boldsymbol{oa''}(\boldsymbol{T}_1\boldsymbol{o}), \boldsymbol{v}, \boldsymbol{a''})$. Then $\boldsymbol{T}_4$ is the closed term we are looking for.

Syllogism

Recall the syllogism rule:

$$\frac{A \to B, B \to C}{A \to C}$$

Notice that:

$$[A \to B]^D \equiv \exists\, \boldsymbol{U}, \boldsymbol{Y}\ \forall\, \boldsymbol{x}, \boldsymbol{v}\, (A_D(\boldsymbol{x}, \boldsymbol{Yxv}, \boldsymbol{a'}) \to B_D(\boldsymbol{Ux}, \boldsymbol{v}, \boldsymbol{a''}))$$
$$[B \to C]^D \equiv \exists\, \boldsymbol{Q}, \boldsymbol{V}\ \forall\, \boldsymbol{u}, \boldsymbol{r}\, (B_D(\boldsymbol{u}, \boldsymbol{Vur}, \boldsymbol{a''}) \to C_D(\boldsymbol{Qu}, \boldsymbol{r}, \boldsymbol{a'''}))$$
$$[A \to C]^D \equiv \exists\, \boldsymbol{Q}, \boldsymbol{Y}\ \forall\, \boldsymbol{x}, \boldsymbol{r}\, (A_D(\boldsymbol{x}, \boldsymbol{Yxr}, \boldsymbol{a'}) \to C_D(\boldsymbol{Qx}, \boldsymbol{r}, \boldsymbol{a'''}))$$

By induction hypothesis, there are terms $\boldsymbol{T}_1, \ldots, \boldsymbol{T}_4$ such that:

$$\forall\, \boldsymbol{x}, \boldsymbol{v}\, (A_D(\boldsymbol{x}, \boldsymbol{T}_2\boldsymbol{a'a''xv}, \boldsymbol{a'}) \to B_D(\boldsymbol{T}_1\boldsymbol{a'a''x}, \boldsymbol{v}, \boldsymbol{a''})) \tag{2.12}$$
$$\forall\, \boldsymbol{u}, \boldsymbol{r}\, (B_D(\boldsymbol{u}, \boldsymbol{T}_4\boldsymbol{a''a'''ur}, \boldsymbol{a''}) \to C_D(\boldsymbol{T}_3\boldsymbol{a''a'''u}, \boldsymbol{r}, \boldsymbol{a'''})) \tag{2.13}$$

Let $\boldsymbol{o}$ be the result of replacing each variable that appears in $\boldsymbol{a''}$ but not in $\boldsymbol{a'}, \boldsymbol{a'''}$ by $\mathscr{O}$ of appropriate type. Take $\boldsymbol{T}_5$ and $\boldsymbol{T}_6$ as:

$$\boldsymbol{T}_5 := \lambda\, \boldsymbol{a'}, \boldsymbol{a'''}, \boldsymbol{x} \,.\, \boldsymbol{T}_3\boldsymbol{oa'''}(\boldsymbol{T}_1\boldsymbol{a'ox})$$
$$\boldsymbol{T}_6 := \lambda\, \boldsymbol{a'}, \boldsymbol{a'''}, \boldsymbol{x}, \boldsymbol{r} \,.\, \boldsymbol{T}_2\boldsymbol{a'ox}(\boldsymbol{T}_4\boldsymbol{oa'''}(\boldsymbol{T}_1\boldsymbol{a'ox})\boldsymbol{r})$$

Instantiate $\boldsymbol{x}$ in (2.12) by itself. Now instantiate $\boldsymbol{u}$ in (2.13) by $\boldsymbol{T}_1\boldsymbol{a}'\boldsymbol{o}\boldsymbol{x}$ and $\boldsymbol{r}$ by itself. Finally, instantiate $\boldsymbol{v}$ in (2.12) by $\boldsymbol{T}_4\boldsymbol{o}\boldsymbol{a}'''(\boldsymbol{T}_1\boldsymbol{a}'\boldsymbol{o}\boldsymbol{x})\boldsymbol{r}$. In the end we obtain:

$$A_D(\boldsymbol{x}, \boldsymbol{T}_2\boldsymbol{a}'\boldsymbol{o}\boldsymbol{x}(\boldsymbol{T}_4\boldsymbol{o}\boldsymbol{a}'''(\boldsymbol{T}_1\boldsymbol{a}'\boldsymbol{o}\boldsymbol{x})\boldsymbol{r}), \boldsymbol{a}') \to B_D(\boldsymbol{T}_1\boldsymbol{a}'\boldsymbol{o}\boldsymbol{x}, \boldsymbol{T}_4\boldsymbol{o}\boldsymbol{a}'''(\boldsymbol{T}_1\boldsymbol{a}'\boldsymbol{o}\boldsymbol{x})\boldsymbol{r}, \boldsymbol{o})$$
$$B_D(\boldsymbol{T}_1\boldsymbol{a}'\boldsymbol{o}\boldsymbol{x}, \boldsymbol{T}_4\boldsymbol{o}\boldsymbol{a}'''(\boldsymbol{T}_1\boldsymbol{a}'\boldsymbol{o}\boldsymbol{x})\boldsymbol{r}, \boldsymbol{o}) \to C_D(\boldsymbol{T}_3\boldsymbol{o}\boldsymbol{a}'''(\boldsymbol{T}_1\boldsymbol{a}'\boldsymbol{o}\boldsymbol{x}), \boldsymbol{r}, \boldsymbol{a}''')$$

By the syllogism rule applied to the previous two expressions, we conclude:

$$A_D(\boldsymbol{x}, \boldsymbol{T}_2\boldsymbol{a}'\boldsymbol{o}\boldsymbol{x}(\boldsymbol{T}_4\boldsymbol{o}\boldsymbol{a}'''(\boldsymbol{T}_1\boldsymbol{a}'\boldsymbol{o}\boldsymbol{x})\boldsymbol{r}), \boldsymbol{a}') \to C_D(\boldsymbol{T}_3\boldsymbol{o}\boldsymbol{a}'''(\boldsymbol{T}_1\boldsymbol{a}'\boldsymbol{o}\boldsymbol{x}), \boldsymbol{r}, \boldsymbol{a}''')$$

Then $\boldsymbol{T}_5$ and $\boldsymbol{T}_6$ are closed terms such that:

$$\forall\, \boldsymbol{x}, \boldsymbol{r}\, (A_D(\boldsymbol{x}, \boldsymbol{T}_6\boldsymbol{a}'\boldsymbol{a}'''\boldsymbol{x}\boldsymbol{r}, \boldsymbol{a}') \to C_D(\boldsymbol{T}_5\boldsymbol{a}'\boldsymbol{a}'''\boldsymbol{x}, \boldsymbol{r}, \boldsymbol{a}'''))$$

Exportation and importation

Recall the exportation and importation rules:

$$\frac{A \wedge B \to C}{A \to B \to C} \qquad \frac{A \to B \to C}{A \wedge B \to C}$$

Notice that:

$$[A \wedge B \to C]^D \equiv \exists\, \boldsymbol{Q}, \boldsymbol{Y}, \boldsymbol{V}\ \forall\, \boldsymbol{x}, \boldsymbol{u}, \boldsymbol{r}\, (A_D(\boldsymbol{x}, \boldsymbol{Y}\boldsymbol{x}\boldsymbol{u}\boldsymbol{r}, \boldsymbol{a}') \wedge B_D(\boldsymbol{u}, \boldsymbol{V}\boldsymbol{x}\boldsymbol{u}\boldsymbol{r}, \boldsymbol{a}'') \to C_D(\boldsymbol{Q}\boldsymbol{x}\boldsymbol{u}, \boldsymbol{r}, \boldsymbol{a}'''))$$
$$[A \to B \to C]^D \equiv \exists\, \boldsymbol{\mathcal{Q}}, \boldsymbol{\mathcal{V}}, \boldsymbol{Y}\ \forall\, \boldsymbol{x}, \boldsymbol{u}, \boldsymbol{r}\, (A_D(\boldsymbol{x}, \boldsymbol{Y}\boldsymbol{x}\boldsymbol{u}\boldsymbol{r}, \boldsymbol{a}') \to B_D(\boldsymbol{u}, \boldsymbol{\mathcal{V}}\boldsymbol{x}\boldsymbol{u}\boldsymbol{r}, \boldsymbol{a}'') \to C_D(\boldsymbol{\mathcal{Q}}\boldsymbol{x}\boldsymbol{u}, \boldsymbol{r}, \boldsymbol{a}'''))$$

As both expressions are equal modulo the importation and exportation rules, and interchanging Q with $\mathcal{Q}$ and V with $\mathcal{V}$, a solution for one is a solution for the other, and we are done.

Expansion

Recall the expansion rule:

$$\frac{A \to B}{C \vee A \to C \vee B}$$

Notice that:

$$[A \to B]^D \equiv \exists\, \boldsymbol{U}, \boldsymbol{Y}\ \forall\, \boldsymbol{x}, \boldsymbol{v}\, (A_D(\boldsymbol{x}, \boldsymbol{Y}\boldsymbol{x}\boldsymbol{v}, \boldsymbol{a}') \to B_D(\boldsymbol{U}\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{a}''))$$
$$[C \vee A \to C \vee B]^D \equiv \exists\, W, \boldsymbol{O}, \boldsymbol{U}, \boldsymbol{R}, \boldsymbol{Y}\ \forall\, z, \boldsymbol{q}, \boldsymbol{x}, \boldsymbol{p}, \boldsymbol{v}$$
$$((z = 0 \to C_D(\boldsymbol{q}, \boldsymbol{R}z\boldsymbol{q}\boldsymbol{x}\boldsymbol{p}\boldsymbol{v}, \boldsymbol{a}''')) \wedge (z \neq 0 \to A_D(\boldsymbol{x}, \boldsymbol{Y}z\boldsymbol{q}\boldsymbol{x}\boldsymbol{p}\boldsymbol{v}, \boldsymbol{a}')) \to$$
$$(Wz\boldsymbol{q}\boldsymbol{x} = 0 \to C_D(\boldsymbol{O}z\boldsymbol{q}\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{a}''')) \wedge (Wz\boldsymbol{q}\boldsymbol{x} \neq 0 \to B_D(\boldsymbol{U}z\boldsymbol{q}\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{a}'')))$$

By induction hypothesis there are terms $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ such that

$$\forall\, \boldsymbol{x}, \boldsymbol{v}\, (A_D(\boldsymbol{x}, \boldsymbol{T}_2\boldsymbol{a}'\boldsymbol{a}''\boldsymbol{x}\boldsymbol{v}, \boldsymbol{a}') \to B_D(\boldsymbol{T}_1\boldsymbol{a}'\boldsymbol{a}''\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{a}''))$$

We need to find terms $T_3, \boldsymbol{T}_4, \ldots, \boldsymbol{T}_7$ such that:

$$\forall\, z, \boldsymbol{q}, \boldsymbol{x}, \boldsymbol{p}, \boldsymbol{v}\, ((z = 0 \to C_D(\boldsymbol{q}, \boldsymbol{T}_6\boldsymbol{a}z\boldsymbol{q}\boldsymbol{x}\boldsymbol{p}\boldsymbol{v}, \boldsymbol{a}''')) \wedge (z \neq 0 \to A_D(\boldsymbol{x}, \boldsymbol{T}_7\boldsymbol{a}z\boldsymbol{q}\boldsymbol{x}\boldsymbol{p}\boldsymbol{v}, \boldsymbol{a}'))$$
$$\to (T_3\boldsymbol{a}z\boldsymbol{q}\boldsymbol{x} = 0 \to C_D(\boldsymbol{T}_4\boldsymbol{a}z\boldsymbol{q}\boldsymbol{x}, \boldsymbol{p}, \boldsymbol{a}''')) \wedge (T_3\boldsymbol{a}z\boldsymbol{q}\boldsymbol{x} \neq 0 \to B_D(\boldsymbol{T}_5\boldsymbol{a}z\boldsymbol{q}\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{a}'')))$$

Let:

$$T_3 := \lambda\, \boldsymbol{a}, z, \boldsymbol{q}, \boldsymbol{x} \,.\, z$$
$$\boldsymbol{T}_4 := \lambda\, \boldsymbol{a}, z, \boldsymbol{q}, \boldsymbol{x} \,.\, \boldsymbol{q}$$
$$\boldsymbol{T}_5 := \lambda\, \boldsymbol{a}, z, \boldsymbol{q}, \boldsymbol{x} \,.\, \boldsymbol{T}_1\boldsymbol{a}'\boldsymbol{a}''\boldsymbol{x}$$
$$\boldsymbol{T}_6 := \lambda\, \boldsymbol{a}, z, \boldsymbol{q}, \boldsymbol{x}, \boldsymbol{p}, \boldsymbol{v} \,.\, \boldsymbol{p}$$
$$\boldsymbol{T}_7 := \lambda\, \boldsymbol{a}, z, \boldsymbol{q}, \boldsymbol{x}, \boldsymbol{p}, \boldsymbol{v} \,.\, \boldsymbol{T}_2\boldsymbol{a}'\boldsymbol{a}''\boldsymbol{x}\boldsymbol{v}$$

Then:

$$\forall\, z, \boldsymbol{q}, \boldsymbol{x}, \boldsymbol{p}, \boldsymbol{v}\, ((z = 0 \to C_D(\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{a}''')) \wedge (z \neq 0 \to A_D(\boldsymbol{x}, \boldsymbol{T}_2\boldsymbol{a}'\boldsymbol{a}''\boldsymbol{x}\boldsymbol{v}, \boldsymbol{a}'))$$
$$\to (z = 0 \to C_D(\boldsymbol{q}, \boldsymbol{p}, \boldsymbol{a}''')) \wedge (z \neq 0 \to B_D(\boldsymbol{T}_1\boldsymbol{a}'\boldsymbol{a}''\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{a}'')))$$

Quantifier rule ($\forall$)

Recall the quantifier rule for $\forall$:

$$\frac{B \to A}{B \to \forall z\, A},\ z \notin \mathrm{fv}(B)$$

Let the free variables of A be in $z, \boldsymbol{a'}$, and the free variables of B be in $\boldsymbol{a''}$ (where z is not one of the a''_i).

Notice that:

$$[B \to A]^D \equiv \exists \boldsymbol{X}, \boldsymbol{V}\ \forall \boldsymbol{u}, \boldsymbol{y}\, (B_D(\boldsymbol{u}, \boldsymbol{V}\boldsymbol{u}\boldsymbol{y}, \boldsymbol{a''}) \to A_D(\boldsymbol{X}\boldsymbol{u}, \boldsymbol{y}, z, \boldsymbol{a'}))$$
$$[B \to \forall z\, A]^D \equiv \exists \boldsymbol{\mathcal{X}}, \boldsymbol{V}\ \forall \boldsymbol{u}, z, \boldsymbol{y}\, (B_D(\boldsymbol{u}, \boldsymbol{V}\boldsymbol{u}z\boldsymbol{y}, \boldsymbol{a''}) \to A_D(\boldsymbol{\mathcal{X}}\boldsymbol{u}z, \boldsymbol{y}, z, \boldsymbol{a'}))$$

By induction hypothesis, we know that there exist terms $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ such that:

$$\forall \boldsymbol{u}, \boldsymbol{y}\, (B_D(\boldsymbol{u}, \boldsymbol{T}_2 z\boldsymbol{a'}\boldsymbol{a''}\boldsymbol{u}\boldsymbol{y}, \boldsymbol{a''}) \to A_D(\boldsymbol{T}_1 z\boldsymbol{a'}\boldsymbol{a''}\boldsymbol{u}, \boldsymbol{y}, z, \boldsymbol{a'})) \tag{2.14}$$

We need to find closed terms $\boldsymbol{T}_3$ and $\boldsymbol{T}_4$ such that:

$$\forall \boldsymbol{u}, z, \boldsymbol{y}\, (B_D(\boldsymbol{u}, \boldsymbol{T}_4\boldsymbol{a'}\boldsymbol{a''}\boldsymbol{u}z\boldsymbol{y}, \boldsymbol{a''}) \to A_D(\boldsymbol{T}_3\boldsymbol{a'}\boldsymbol{a''}\boldsymbol{u}z, \boldsymbol{y}, z, \boldsymbol{a'})) \tag{2.15}$$

is provable in WE-HA$^\omega$. Let:

$$\boldsymbol{T}_3 := \lambda\, \boldsymbol{a'}, \boldsymbol{a''}, \boldsymbol{u}, z\, .\, \boldsymbol{T}_1 z\boldsymbol{a'}\boldsymbol{a''}\boldsymbol{u}$$
$$\boldsymbol{T}_4 := \lambda\, \boldsymbol{a'}, \boldsymbol{a''}, \boldsymbol{u}, z, \boldsymbol{y}\, .\, \boldsymbol{T}_2 z\boldsymbol{a'}\boldsymbol{a''}\boldsymbol{u}\boldsymbol{y}$$

Then (2.15) reduces to:

$$\forall \boldsymbol{u}, z, \boldsymbol{y}\, (B_D(\boldsymbol{u}, \boldsymbol{T}_2 z\boldsymbol{a'}\boldsymbol{a''}\boldsymbol{u}\boldsymbol{y}, \boldsymbol{a''}) \to A_D(\boldsymbol{T}_1 z\boldsymbol{a'}\boldsymbol{a''}\boldsymbol{u}, \boldsymbol{y}, z, \boldsymbol{a'}))$$

This can be proved using (2.14), the induction hypothesis: first instantiate $\boldsymbol{u}$ by itself, then generalize at z and finally generalize at $\boldsymbol{u}$.

Quantifier rule ($\exists$)

Recall the quantifier rule for $\exists$:

$$\frac{A \to B}{\exists z\, A \to B},\ z \notin \mathrm{fv}(B)$$

Let the free variables of A be in $z, \boldsymbol{a'}$, and the free variables of B be in $\boldsymbol{a''}$ (where z is not one of the a''_i).

Notice that:

$$[A \to B]^D \equiv \exists \boldsymbol{U}, \boldsymbol{Y}\ \forall \boldsymbol{x}, \boldsymbol{v}\, (A_D(\boldsymbol{x}, \boldsymbol{Y}\boldsymbol{x}\boldsymbol{v}, z, \boldsymbol{a'}) \to B_D(\boldsymbol{U}\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{a''}))$$
$$[\exists z\, A \to B]^D \equiv \exists \boldsymbol{U}, \boldsymbol{Y}\ \forall z, \boldsymbol{x}, \boldsymbol{v}\, (A_D(\boldsymbol{x}, \boldsymbol{Y}z\boldsymbol{x}\boldsymbol{v}, z, \boldsymbol{a'}) \to B_D(\boldsymbol{U}z\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{a''}))$$

By induction hypothesis, there are terms $\boldsymbol{T}_1$ and $\boldsymbol{T}_2$ such that:

$$\forall \boldsymbol{x}, \boldsymbol{v}\, (A_D(\boldsymbol{x}, \boldsymbol{T}_2 z\boldsymbol{a'}\boldsymbol{a''}\boldsymbol{x}\boldsymbol{v}, z, \boldsymbol{a'}) \to B_D(\boldsymbol{T}_1 z\boldsymbol{a'}\boldsymbol{a''}\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{a''})) \tag{2.16}$$

We need to find closed terms $\boldsymbol{T}_3$ and $\boldsymbol{T}_4$ such that

$$\forall z, \boldsymbol{x}, \boldsymbol{v}\, (A_D(\boldsymbol{x}, \boldsymbol{T}_4\boldsymbol{a'}\boldsymbol{a''}z\boldsymbol{x}\boldsymbol{v}, z, \boldsymbol{a'}) \to B_D(\boldsymbol{T}_3\boldsymbol{a'}\boldsymbol{a''}z\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{a''}))$$

is provable in WE-HA$^\omega$. Take:

$$\boldsymbol{T}_3 := \lambda\, \boldsymbol{a'}, \boldsymbol{a''}, z, \boldsymbol{x}\, .\, \boldsymbol{T}_1 z\boldsymbol{a'}\boldsymbol{a''}\boldsymbol{x}$$
$$\boldsymbol{T}_4 := \lambda\, \boldsymbol{a'}, \boldsymbol{a''}, z, \boldsymbol{x}, \boldsymbol{v}\, .\, \boldsymbol{T}_2 z\boldsymbol{a'}\boldsymbol{a''}\boldsymbol{x}\boldsymbol{v}$$

These terms do the job. Then we only need to generalize the induction hypothesis (2.16) over z.

$=_0$ and S

The axioms for type 0 equality and for the successor are composed of only prime formulas, $\wedge$ and $\to$. Hence, by Remark 2.2.2, they remain unchanged after the Gödel translation is performed. In other words, if A is one of these axioms, the formula $\forall \boldsymbol{y}\, A_D(\boldsymbol{Ta}, \boldsymbol{y}, \boldsymbol{a})$ is none other than A itself, and to prove it in WE-HA$^\omega$ a single application of that axiom suffices.

Quantifier-free extensionality rule

Recall the quantifier-free extensionality rule, without the abbreviation for higher type equality:

$$\frac{A_0 \to \forall \boldsymbol{z}\,(s\boldsymbol{z} =_0 t\boldsymbol{z})}{A_0 \to \forall \boldsymbol{w}\,(r[s/x]\boldsymbol{w} =_0 r[t/x]\boldsymbol{w})}$$

where $\boldsymbol{z}$ and $\boldsymbol{w}$ are of the appropriate types.

Notice that, by Proposition 1.22, A_0 can be written without $\vee$. Hence, by Remark 2.2.2, $(A_0)^D \equiv A_0$. By Remark 2.2.3, Gödel's translation of the purely universal formulas doesn't change them either. Then:

$$[A_0 \to \forall \boldsymbol{z}\,(s\boldsymbol{z} =_0 t\boldsymbol{z})]^D \equiv \forall \boldsymbol{z}\,(A_0 \to (s\boldsymbol{z} =_0 t\boldsymbol{z}))$$
$$[A_0 \to \forall \boldsymbol{w}\,(r[s/x]\boldsymbol{w} =_0 r[t/x]\boldsymbol{w})]^D \equiv \forall \boldsymbol{w}\,(A_0 \to (r[s/x]\boldsymbol{w} =_0 r[t/x]\boldsymbol{w}))$$

Noticing that $\forall x\,(A \to B(x)) \leftrightarrow (A \to \forall x\, B(x))$ is an intuitionistic truth (as long as $x \notin \mathrm{fv}(A)$):

$$[A_0 \to \forall \boldsymbol{z}\,(s\boldsymbol{z} =_0 t\boldsymbol{z})]^D \leftrightarrow A_0 \to \forall \boldsymbol{z}\,(s\boldsymbol{z} =_0 t\boldsymbol{z})$$
$$[A_0 \to \forall \boldsymbol{w}\,(r[s/x]\boldsymbol{w} =_0 r[t/x]\boldsymbol{w})]^D \leftrightarrow A_0 \to \forall \boldsymbol{w}\,(r[s/x]\boldsymbol{w} =_0 r[t/x]\boldsymbol{w})$$

And we prove the desired using the quantifier-free extensionality rule itself.

Induction schema

For simplicity, we will use the induction rule instead of the schema. Notice that they are equivalent, so this is not a problem.

Suppose that the free variables of A are in $z^0, \boldsymbol{a'}$. Recall the induction rule:

$$\frac{A(0, \boldsymbol{a'}), A(z, \boldsymbol{a'}) \to A(Sz, \boldsymbol{a'})}{A(z, \boldsymbol{a'})}$$

Notice that:

$$[A(0, \boldsymbol{a'})]^D \equiv \exists \boldsymbol{x}\, \forall \boldsymbol{y}\, A_D(\boldsymbol{x}, \boldsymbol{y}, 0, \boldsymbol{a'})$$
$$[A(z, \boldsymbol{a'}) \to A(Sz, \boldsymbol{a'})]^D \equiv \exists \boldsymbol{U}, \boldsymbol{Y}\, \forall \boldsymbol{x}, \boldsymbol{v}\,(A_D(\boldsymbol{x}, \boldsymbol{Yxv}, z, \boldsymbol{a'}) \to A_D(\boldsymbol{Ux}, \boldsymbol{v}, Sz, \boldsymbol{a'}))$$
$$[A(z, \boldsymbol{a'})]^D \equiv \exists \boldsymbol{x}\, \forall \boldsymbol{y}\, A_D(\boldsymbol{x}, \boldsymbol{y}, z, \boldsymbol{a'})$$

By induction hypothesis, there are closed terms $\boldsymbol{T}_1, \boldsymbol{T}_2$ and $\boldsymbol{T}_3$ such that:

$$\forall \boldsymbol{y}\, A_D(\boldsymbol{T}_1\boldsymbol{a'}, \boldsymbol{y}, 0, \boldsymbol{a'}) \tag{2.17}$$
$$\forall \boldsymbol{x}, \boldsymbol{v}\,(A_D(\boldsymbol{x}, \boldsymbol{T}_3 z\boldsymbol{a'}\boldsymbol{x}\boldsymbol{v}, z, \boldsymbol{a'}) \to A_D(\boldsymbol{T}_2 z\boldsymbol{a'}\boldsymbol{x}, \boldsymbol{v}, Sz, \boldsymbol{a'})) \tag{2.18}$$

Choose $\boldsymbol{T}_4$ such that:

$$\begin{cases} \boldsymbol{T}_4 0\boldsymbol{a'} = \boldsymbol{T}_1\boldsymbol{a'} \\ \boldsymbol{T}_4(Sz)\boldsymbol{a'} = \boldsymbol{T}_2 z\boldsymbol{a'}(\boldsymbol{T}_4 z\boldsymbol{a'}) \end{cases}$$

This is possible using the simultaneous recursors.

We need to prove:

$$\forall \boldsymbol{y}\, A_D(\boldsymbol{T}_4 z\boldsymbol{a'}, \boldsymbol{y}, z, \boldsymbol{a'})$$

We will do it using the induction rule. So, in order to conclude the desired, we first need to show that:

$$\begin{cases} \forall \boldsymbol{y}\, A_D(\boldsymbol{T}_4 0\boldsymbol{a'}, \boldsymbol{y}, 0, \boldsymbol{a'}) \\ \forall \boldsymbol{y}\, A_D(\boldsymbol{T}_4 z\boldsymbol{a'}, \boldsymbol{y}, z, \boldsymbol{a'}) \to \forall \boldsymbol{y}\, A_D(\boldsymbol{T}_4(Sz)\boldsymbol{a'}, \boldsymbol{y}, Sz, \boldsymbol{a'}) \end{cases}$$

Using the definition of $\boldsymbol{T}_4$, it suffices to show that:

$$\begin{cases} \forall\, \boldsymbol{y}\, A_D(\boldsymbol{T}_1\boldsymbol{a'}, \boldsymbol{y}, 0, \boldsymbol{a'}) \\ \forall\, \boldsymbol{y}\, A_D(\boldsymbol{T}_4 z\boldsymbol{a'}, \boldsymbol{y}, z, \boldsymbol{a'}) \to \forall\, \boldsymbol{y}\, A_D(\boldsymbol{T}_2 z\boldsymbol{a'}(\boldsymbol{T}_4 z\boldsymbol{a'}), \boldsymbol{y}, Sz, \boldsymbol{a'}) \end{cases}$$

The first line coincides with our first induction hypothesis (2.17). For the second, instantiate $\boldsymbol{x}$ by $\boldsymbol{T}_4 z\boldsymbol{a'}$ and $\boldsymbol{v}$ by itself in (2.18), obtaining:

$$A_D(\boldsymbol{T}_4 z\boldsymbol{a'}, \boldsymbol{T}_3 z\boldsymbol{a'}(\boldsymbol{T}_4 z\boldsymbol{a'})\boldsymbol{v}, z, \boldsymbol{a'}) \to A_D(\boldsymbol{T}_2 z\boldsymbol{a'}(\boldsymbol{T}_4 z\boldsymbol{a'}), \boldsymbol{v}, Sz, \boldsymbol{a'}) \tag{2.19}$$

Assume $\forall\, \boldsymbol{y}\, A_D(\boldsymbol{T}_4 z\boldsymbol{a'}, \boldsymbol{y}, z, \boldsymbol{a'})$. Then we can instantiate $\boldsymbol{y}$ by $\boldsymbol{T}_3 z\boldsymbol{a'}(\boldsymbol{T}_4 z\boldsymbol{a'})\boldsymbol{v}$, obtaining the antecedent of (2.19). Thus we con conclude its consequent, and, generalizing on $\boldsymbol{v}$, we have:

$$\forall\, \boldsymbol{v}\, A_D(\boldsymbol{T}_2 z\boldsymbol{a'}(\boldsymbol{T}_4 z\boldsymbol{a'}), \boldsymbol{v}, Sz, \boldsymbol{a'})$$

as we wanted to show.

$\boxed{\Pi, \Sigma, \boldsymbol{R}}$

The axioms for Π, Σ and $\boldsymbol{R}$ are all higher type equalities between prime formulas. Hence they are composed of universal quantifications followed by a prime formula, and by Remark 2.2.3, their Gödel's translations are themselves, and can be proved by themselves. There are no existentially quantified variables, so there is no need for witnesses.

$\boxed{\mathrm{AC}^\omega}$

Recall the schema of choice:

$$\forall\, z\; \exists\, w\, A(z, w) \to \exists\, W\; \forall\, z\, A(z, Wz)$$

The Gödel translations of the antecedent and consequent are the same:

$$\begin{aligned} [\forall\, z\; \exists\, w\, A(z, w, \boldsymbol{a'})]^D &\equiv [\forall\, z\; \exists\, w\; \exists\, \boldsymbol{x}\; \forall\, \boldsymbol{y}\, A_D(\boldsymbol{x}, \boldsymbol{y}, z, w, \boldsymbol{a'})]^D \\ &\equiv \exists\, W, \boldsymbol{X}\; \forall\, z, \boldsymbol{y}\, A_D(\boldsymbol{X}z, \boldsymbol{y}, z, Wz, \boldsymbol{a'}) \\ [\exists\, W\; \forall\, z\, A(z, Wz)]^D &\equiv [\exists\, W\; \forall\, z\; \exists\, \boldsymbol{x}\; \forall\, \boldsymbol{y}\, A_D(\boldsymbol{x}, \boldsymbol{y}, z, Wz, \boldsymbol{a'})]^D \\ &\equiv \exists\, W, \boldsymbol{X}\; \forall\, z, \boldsymbol{y}\, A_D(\boldsymbol{X}z, \boldsymbol{y}, z, Wz, \boldsymbol{a'}) \end{aligned}$$

So, as $[B \to C]^D \equiv [B^D \to C^D]^D$ by Remark 2.2.1, the only thing we actually need in this step is to know that the soundness theorem holds for $D \to D$, for any formula D.
Notice that:

$$[D \to D]^D \equiv \exists\, \boldsymbol{U}, \boldsymbol{Y}\; \forall\, \boldsymbol{x}, \boldsymbol{v}\, (D_D(\boldsymbol{x}, \boldsymbol{Y}\boldsymbol{x}\boldsymbol{v}, \boldsymbol{a}) \to D_D(\boldsymbol{U}\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{a}))$$

Choosing

$$\begin{aligned} \boldsymbol{T_U} &:= \lambda\, \boldsymbol{a}, \boldsymbol{x}\, .\, \boldsymbol{x} \\ \boldsymbol{T_Y} &:= \lambda\, \boldsymbol{a}, \boldsymbol{x}, \boldsymbol{v}\, .\, \boldsymbol{v} \end{aligned}$$

clearly does the job.

$\boxed{\mathrm{M}^\omega}$

Recall Markov's principle:

$$(\forall\, \boldsymbol{z}\, A_0(\boldsymbol{z}) \to \perp) \to \exists\, \boldsymbol{z}\, (A_0(\boldsymbol{z}) \to \perp)$$

Notice that, since A_0 is a quantifier-free formula and hence can, by Corollary 1.22, be written without $\vee$, we know by Remark 2.2, that the Gödel translation of $\forall\, \boldsymbol{z}\, A_0(\boldsymbol{z})$ and $A_0(\boldsymbol{z}) \to \perp$ doesn't change either of those formulas. So:

$$\begin{aligned} [(\forall\, \boldsymbol{z}\, A_0(\boldsymbol{z}) \to \perp)]^D &\equiv \exists\, \boldsymbol{z}\, (A_0(\boldsymbol{z}) \to \perp) \\ [\exists\, \boldsymbol{z}\, (A_0(\boldsymbol{z}) \to \perp)]^D &\equiv \exists\, \boldsymbol{z}\, (A_0(\boldsymbol{z}) \to \perp) \end{aligned}$$

and we are again in the case where proving the soundness of $D \to D$ suffices (see the step for AC^ω).

$\boxed{\mathrm{IP}^{\omega}_{\forall}}$

Recall the independence of premise schema for purely universal premises:

$$(\forall \boldsymbol{z}\, A_0(\boldsymbol{z}) \to \exists\, w\, B(w)) \to \exists\, w\, (\forall \boldsymbol{z}\, A_0(\boldsymbol{z}) \to B(w))$$

Notice that:

$$[\forall \boldsymbol{z}\, A_0(\boldsymbol{z}) \to \exists\, w\, B(w)]^D \equiv \exists\, w, \boldsymbol{x}, \boldsymbol{Z}\, \forall \boldsymbol{y}\, (A_0(\boldsymbol{Zy}) \to B_D(\boldsymbol{x}, \boldsymbol{y}, w))$$
$$[\exists\, w\, (\forall \boldsymbol{z}\, A_0(\boldsymbol{z}) \to B(w))]^D \equiv \exists\, w, \boldsymbol{x}, \boldsymbol{Z}\, \forall \boldsymbol{y}\, (A_0(\boldsymbol{Zy}) \to B_D(\boldsymbol{x}, \boldsymbol{y}, w))$$

and we are yet again in the case where proving the soundness of $D \to D$ suffices (see the step for AC^{ω}).

$\boxed{\mathcal{P}}$

As $\mathcal{P}$ only contains purely universal formulas, and by Remark 2.2.3 the Gödel translation doesn't change these formulas, there is nothing to be done: they prove their own translation.

□

Theorem 2.8 (Characterisation Theorem)**.** For all formulas A of WE-HA$^{\omega}$:

$$\text{WE-HA}^{\omega} + \mathrm{AC}^{\omega} + \mathrm{M}^{\omega} + \mathrm{IP}^{\omega}_{\forall} \vdash A^D \leftrightarrow A$$

Proof. The proof follows by induction on the logical structure of A. We won't do every detail of the proof: we will simply give an overview and state witch rules are necessary for each step.

$\boxed{A \text{ is a prime formula}}$

Follows directly form the fact that $A^D \equiv A$ and $A \leftrightarrow A$.

$\boxed{A \wedge B}$

By induction hypothesis, $A^D \leftrightarrow A$ and $B^D \leftrightarrow B$, so all we need to show is that:

$$[A \wedge B]^D \leftrightarrow A^D \wedge B^D$$
$$\textit{i.e.}$$
$$\exists\, \boldsymbol{x}, \boldsymbol{u}\, \forall \boldsymbol{y}, \boldsymbol{v}\, (A_D(\boldsymbol{x}, \boldsymbol{v}) \wedge B_D(\boldsymbol{u}, \boldsymbol{v})) \leftrightarrow \exists\, \boldsymbol{x}\, \forall \boldsymbol{y}\, A_D(\boldsymbol{x}, \boldsymbol{v}) \wedge \exists\, \boldsymbol{u}\, \forall \boldsymbol{v}\, B_D(\boldsymbol{u}, \boldsymbol{v})$$

This is a straightforward verification in both directions, since $\boldsymbol{x}, \boldsymbol{y} \notin \mathrm{fv}(B_D)$ and $\boldsymbol{u}, \boldsymbol{v} \notin \mathrm{fv}(A_D)$.

$\boxed{A \vee B}$

Following the same reasoning as in the last step, we need to show:

$$\exists\, z^0, \boldsymbol{x}, \boldsymbol{u}\, \forall \boldsymbol{y}, \boldsymbol{v}\, ((z = 0 \to A_D(\boldsymbol{x}, \boldsymbol{y})) \wedge (z \neq 0 \to B_D(\boldsymbol{u}, \boldsymbol{v}))) \leftrightarrow \exists\, \boldsymbol{x}\, \forall \boldsymbol{y}\, A_D(\boldsymbol{x}, \boldsymbol{y}) \vee \exists\, \boldsymbol{u}\, \forall \boldsymbol{v}\, B_D(\boldsymbol{u}, \boldsymbol{v})$$

From left to right, one needs to use the fact that $z = 0 \vee z \neq 0$ is provable in WE-HA$^{\omega}$ (Corollary 1.21).

From right to left, one needs to chose a suitable term to witness the statement $\exists\, z^0 \cdots$. Simply choosing $z := 0$ when considering A_D and $z := S0$ when considering B_D will suffice.

$\boxed{\exists\, z\, A(z)}$

We want to show:

$$\exists\, z, \boldsymbol{x}\, \forall \boldsymbol{y}\, A_D(\boldsymbol{x}, \boldsymbol{y}, z) \leftrightarrow \exists\, z\, \exists\, \boldsymbol{x}\, \forall \boldsymbol{y}\, A_D(\boldsymbol{x}, \boldsymbol{y}, z)$$

which is obvious, since the only difference between the formulas is the notation used to represent consecutive existential quantifications.

$\boxed{\forall z\, A(z)}$

We want to show:

$$\exists \boldsymbol{X}\ \forall z, \boldsymbol{y}\, A_D(\boldsymbol{X}z, \boldsymbol{y}, z) \leftrightarrow \forall z\ \exists \boldsymbol{x}\ \forall \boldsymbol{y}\, A_D(\boldsymbol{x}, \boldsymbol{y}, z)$$

From left to right, one eventually needs to find a witness for the statement $\exists \boldsymbol{x} \cdots$. Simply chose $\boldsymbol{x} := \boldsymbol{X}z$.

From right to left, $|\boldsymbol{x}|$ applications of AC^ω suffice.

$\boxed{A \to B}$

This is the most complicated step of the proof, and the only one that requires Markov's principle and the independence of premise schema. We want to show:

$$\exists \boldsymbol{U}, \boldsymbol{Y}\ \forall \boldsymbol{x}, \boldsymbol{v}\, (A_D(\boldsymbol{x}, \boldsymbol{Y}\boldsymbol{x}\boldsymbol{v}) \to B_D(\boldsymbol{U}\boldsymbol{x}, \boldsymbol{v})) \leftrightarrow (\exists \boldsymbol{x}\ \forall \boldsymbol{y}\, A_D(\boldsymbol{x}, \boldsymbol{y}) \to \exists \boldsymbol{u}\ \forall \boldsymbol{v}\, B_D(\boldsymbol{u}, \boldsymbol{v}))$$

Consider the following steps:

(1) $\exists \boldsymbol{x}\ \forall \boldsymbol{y}\, A_D(\boldsymbol{x}, \boldsymbol{y}) \to \exists \boldsymbol{u}\ \forall \boldsymbol{v}\, B_D(\boldsymbol{u}, \boldsymbol{v})$

(2) $\forall \boldsymbol{x}\, [\forall \boldsymbol{y}\, A_D(\boldsymbol{x}, \boldsymbol{y}) \to \exists \boldsymbol{u}\ \forall \boldsymbol{v}\, B_D(\boldsymbol{u}, \boldsymbol{v})]$

(3) $\forall \boldsymbol{x}\ \exists \boldsymbol{u}\, [\forall \boldsymbol{y}\, A_D(\boldsymbol{x}, \boldsymbol{y}) \to \forall \boldsymbol{v}\, B_D(\boldsymbol{u}, \boldsymbol{v})]$

(4) $\forall \boldsymbol{x}\ \exists \boldsymbol{u}\ \forall \boldsymbol{v}\, [\forall \boldsymbol{y}\, A_D(\boldsymbol{x}, \boldsymbol{y}) \to B_D(\boldsymbol{u}, \boldsymbol{v})]$

(5) $\forall \boldsymbol{x}\ \exists \boldsymbol{u}\ \forall \boldsymbol{v}\ \exists \boldsymbol{y}\, [A_D(\boldsymbol{x}, \boldsymbol{y}) \to B_D(\boldsymbol{u}, \boldsymbol{v})]$

(6) $\exists \boldsymbol{U}\ \forall \boldsymbol{x}\ \forall \boldsymbol{v}\ \exists \boldsymbol{y}\, [A_D(\boldsymbol{x}, \boldsymbol{y}) \to B_D(\boldsymbol{U}\boldsymbol{x}, \boldsymbol{v})]$

(7) $\exists \boldsymbol{U}\ \exists \boldsymbol{Y}\ \forall \boldsymbol{x}\ \forall \boldsymbol{v}\, [A_D(\boldsymbol{x}, \boldsymbol{Y}\boldsymbol{x}\boldsymbol{v}) \to B_D(\boldsymbol{U}\boldsymbol{x}, \boldsymbol{v})]$

Clearly it is enough to show $(i) \leftrightarrow (i+1)$, $i \in \{1, \ldots, 6\}$. All of the implications $(i+1) \to (i)$ are easy-to-prove intuitionistic truths, as are (1) $\to$ (2) and (3) $\to$ (4). Both (5) $\to$ (6) and (6) $\to$ (7) are direct applications of AC^ω. The implication (2) $\to$ (3) is justified by $\mathrm{IP}^\omega_\forall$, noting that $\forall \boldsymbol{y}\, A_D(\boldsymbol{x}, \boldsymbol{y})$ is a purely universal formula. Finally, (4) $\to$ (5) is a not-so-direct consequence of the Markov principle; we basically need to show that:

$$(\forall \boldsymbol{y}\, A_0(\boldsymbol{y}) \to B_0) \to \exists \boldsymbol{y}\, (A_0(\boldsymbol{y}) \to B_0)$$

for all quantifier-free formulas A_0, B_0 such that $\boldsymbol{y} \notin \mathrm{fv}(B_0)$.

Taking into consideration that B_0 is quantifier-free and Corollary 1.21, there are two cases to consider: either B_0, or $\neg B_0$. Suppose B_0. Then $A_0(\boldsymbol{y}) \to B_0$ clearly follows, and hence $\exists \boldsymbol{y}\, (A_0(\boldsymbol{y}) \to B_0)$. Suppose now that we have $\neg B_0$, and assume $\forall \boldsymbol{y}\, A_0(\boldsymbol{y}) \to B_0$. It is intuitionistically the case that $(C \to D) \to (\neg D \to \neg C)$ for any formulas C, D, so from $\forall \boldsymbol{y}\, A_0(\boldsymbol{y}) \to B_0$ and $\neg B_0$ we conclude $\neg \forall \boldsymbol{y}\, A_0(\boldsymbol{y})$. By Markov's principle we get $\exists \boldsymbol{y}\, \neg A_0(\boldsymbol{y})$. Let $\boldsymbol{y_0}$ be a witness to that statement, that is to say, $\boldsymbol{y_0}$ is such that $\neg A_0(\boldsymbol{y_0})$. Then it is clear that $A_0(\boldsymbol{y_0}) \to B_0$ and we are finally able to conclude $\exists \boldsymbol{y}\, (A_0(\boldsymbol{y}) \to B_0)$.

□

3 Majorizability and the Monotone Functional Interpretation

In this section we aim to state and prove the soundness theorem for another functional interpretation: the monotone functional interpretation, due to Kohlenbach ([Kohlenbach, 1996]). In 3.1 we give the preliminary definitions and prove Howard's majorization theorem. In 3.2 we explore the monotone functional interpretation.

3.1 Majorizability

We wish to define a "less than" predicate between type 0 terms ($<_0$) and a "maximum" term ($\max_0^{0\to0\to0}$) in WE-HA^ω. It is possible to define both (in order to define a new predicate, define instead its "characteristic term" - a closed term that is equal to 0 if and only if the predicate holds) such that the following lemma is provable:

Lemma 3.1 (Axioms for $<_0$ and $\max_0$)**.**

1. $x_1 =_0 x_2 \wedge y_1 =_0 y_2 \wedge x_1 <_0 y_1 \to x_2 <_0 y_2$;
2. $\neg(x <_0 0)$;
3. $x <_0 y \vee x =_0 y \vee x >_0 y$;
4. $x <_0 Sy \leftrightarrow x \leq_0 y$;
5. $x <_0 y \to Sx <_0 Sy$;
6. $x <_0 y \wedge y <_0 z \to x <_0 z$;
7. $\max_0 xy \geq_0 x$;
8. $\max_0 xy \geq_0 y$;
9. $\max_0 xy =_0 x \vee \max_0 xy =_0 y$.

We further extend $<_0$ to complex types as:

$$x <_{\rho\to\tau} y :\equiv \forall r^\rho \, (xr <_\tau yr)$$

and define the following useful abbreviations:

- $x \leq_\rho y :\equiv x <_\rho y \vee x =_\rho y$;
- $x >_\rho y :\equiv y <_\rho x$;
- $x \geq_\rho y :\equiv y <_\rho x \vee x =_\rho y$.

We also define the maximum for complex types as:

$$\max_{\rho\to\tau} xy := \lambda r^\rho . \max_\tau(xr)(yr)$$

Definition 3.2 (s-maj)**.** We define strong majorizability x^* s-maj$_\rho$ x between terms of type ρ by induction on the type:

$$\begin{aligned} &x^* \text{ s-maj}_0\, x :\equiv x^* \geq_0 x \\ &x^* \text{ s-maj}_{\rho\to\tau}\, x :\equiv \forall r, r^* \, (r^* \text{ s-maj}_\rho\, r \to x^* r^* \text{ s-maj}_\tau\, xr \wedge x^* r^* \text{ s-maj}_\tau\, x^* r) \end{aligned}$$

Lemma 3.3. WE-HA$^\omega$ proves:

1. $x =_\rho y \wedge x^* =_\rho y^* \wedge x^* \text{ s-maj}_\rho\, x \to y^* \text{ s-maj}_\rho\, y$;
2. $x^* \text{ s-maj}_\rho\, x \to x^* \text{ s-maj}_\rho\, x^*$;
3. $x \text{ s-maj}_\rho\, y \wedge y \text{ s-maj}_\rho\, z \to x \text{ s-maj}_\rho\, z$;
4. $x^* \text{ s-maj}_\rho\, x \wedge x \geq_\rho y \to x^* \text{ s-maj}_\rho\, y$;
5. For $\rho = \rho_1 \to \cdots \to \rho_k \to \tau$:

$$x^* \text{ s-maj}_\rho\, x \leftrightarrow \forall r_1, r_1^*, \ldots, r_k, r_k^* \left(\bigwedge_{i=0}^{k} (r_i^* \text{ s-maj}_{\rho_i}\, r_i) \to x^* \boldsymbol{r^*} \text{ s-maj}_\tau\, x\boldsymbol{r} \wedge x^* \boldsymbol{r^*} \text{ s-maj}_\tau\, x^* \boldsymbol{r} \right)$$

Proof of 1. By induction on the type:

$\boxed{0}$ Direct from Lemma 3.1.1 and the quantifier-free extensionality rule.

$\boxed{\rho \to \tau}$

Assume the following:

$$x =_{\rho\to\tau} y \tag{3.1}$$
$$x^* =_{\rho\to\tau} y^* \tag{3.2}$$
$$x^* \text{ s-maj}_{\rho\to\tau}\, x \equiv \forall r, r^* \, (r^* \text{ s-maj}_\rho\, r \to x^* r^* \text{ s-maj}_\tau\, xr \wedge x^* r^* \text{ s-maj}_\tau\, x^* r) \tag{3.3}$$

We wish to prove:

$$y^* \text{ s-maj}_{\rho\to\tau}\, y \equiv \forall r, r^* \, (r^* \text{ s-maj}_\rho\, r \to y^* r^* \text{ s-maj}_\tau\, yr \wedge y^* r^* \text{ s-maj}_\tau\, y^* r)$$

Take any r, r^* of the appropriate types, such that r^* s-maj$_\rho$ r.

Notice that:

(i) From (3.1) and the definition of $=_{\rho\to\tau}$ we have $xr =_\tau yr$;

(ii) From (3.2) and the definition of $=_{\rho\to\tau}$ we have $x^*r^* =_\tau y^*r^*$;

(iii) From (3.2) and the definition of $=_{\rho\to\tau}$ we have $x^*r =_\tau y^*r$.

By induction hypothesis, (3.3), (i) and (ii) we conclude $y^*r^* \text{ s-maj}_\tau\, yr$. By induction hypothesis, (3.3), (ii) and (iii) we conclude $y^*r^* \text{ s-maj}_\tau\, y^*r$.

□

Proof of 2. We differentiate between 0 and complex types:

$\boxed{0}$ Direct from the fact that $x^* \geq_0 x^*$.

$\boxed{\rho \to \tau}$

Notice that:

$$\begin{aligned} x^* \text{ s-maj}_{\rho\to\tau}\, x &\equiv \forall r, r^* \,(r^* \text{ s-maj}_\rho\, r \to x^*r^* \text{ s-maj}_\tau\, xr \wedge x^*r^* \text{ s-maj}_\tau\, x^*r) \\ &\to \forall r, r^* \,(r^* \text{ s-maj}_\rho\, r \to x^*r^* \text{ s-maj}_\tau\, x^*r) \\ &\leftrightarrow \forall r, r^* \,(r^* \text{ s-maj}_\rho\, r \to x^*r^* \text{ s-maj}_\tau\, x^*r \wedge x^*r^* \text{ s-maj}_\tau\, x^*r) \\ &\equiv x^* \text{ s-maj}_{\rho\to\tau}\, x^* \end{aligned}$$

and we are done.

□

Proof of 3. By induction on the type:

$\boxed{0}$ Direct from Lemma 3.1.6 and the transitivity of equality.

$\boxed{\rho \to \tau}$

Our hypothesis are:

$$x \text{ s-maj}_{\rho\to\tau}\, y \equiv \forall r, r^* \,(r^* \text{ s-maj}_\rho\, r \to xr^* \text{ s-maj}_\tau\, yr \wedge xr^* \text{ s-maj}_\tau\, xr) \tag{3.4}$$

$$y \text{ s-maj}_{\rho\to\tau}\, z \equiv \forall r, r^* \,(r^* \text{ s-maj}_\rho\, r \to yr^* \text{ s-maj}_\tau\, zr \wedge yr^* \text{ s-maj}_\tau\, yr) \tag{3.5}$$

And we want to show:

$$x \text{ s-maj}_{\rho\to\tau}\, z \equiv \forall s, s^* \,(s^* \text{ s-maj}_\rho\, s \to xs^* \text{ s-maj}_\tau\, zs \wedge xs^* \text{ s-maj}_\tau\, xs)$$

Taking arbitrary s, s^* such that $s^* \text{ s-maj}_\rho\, s$, we need:

$$xs^* \text{ s-maj}_\tau\, zs \tag{3.6}$$

$$xs^* \text{ s-maj}_\tau\, xs \tag{3.7}$$

Noticing that from $s^* \text{ s-maj}_\rho\, s$ and 2, $s^* \text{ s-maj}_\rho\, s^*$:

- $xs^* \text{ s-maj}_\tau\, ys^*$, from (3.4), instantiating $r := s^*$ and $r^* := s^*$;
- $ys^* \text{ s-maj}_\tau\, zs$, from (3.5), instantiating $r := s$ and $r^* := s^*$;

(3.6) follows by induction hypothesis in type τ.

As for (3.7), it is a consequence of (3.4), instantiating $r := s$ and $r^* := s^*$. □

Proof of 4. By induction on the type:

$\boxed{0}$ Direct from Lemma 3.1.6 and the transitivity of equality.

$\boxed{\rho \to \tau}$

Our hypothesis are:

$$x^* \text{ s-maj}_{\rho\to\tau}\, x \equiv \forall\, r, r^* \, (r^* \text{ s-maj}_\rho\, r \to x^* r^* \text{ s-maj}_\tau\, xr \wedge x^* r^* \text{ s-maj}_\tau\, x^* r) \tag{3.8}$$

$$x \geq_{\rho\to\tau} y \equiv \forall\, r^\rho\, xr \geq_\tau yr \tag{3.9}$$

And we want to show:

$$x^* \text{ s-maj}_{\rho\to\tau}\, y \equiv \forall\, r, r^* \, (r^* \text{ s-maj}_\rho\, r \to x^* r^* \text{ s-maj}_\tau\, yr \wedge x^* r^* \text{ s-maj}_\tau\, x^* r)$$

Taking arbitrary r, r^* such that $r^* \text{ s-maj}_\rho\, r$, we need:

$$x^* r^* \text{ s-maj}_\tau\, yr \tag{3.10}$$

$$x^* r^* \text{ s-maj}_\tau\, x^* r \tag{3.11}$$

Noting that:

- $x^* r^* \text{ s-maj}_\tau\, xr$, from (3.8);
- $xr \geq_\tau yr$, from (3.9);

(3.10) follows by induction hypothesis in type τ.

As for (3.11), it is a consequence of (3.8).

□

Proof of 5. By induction on k:

$\boxed{k = 1}$ This reduces to the definition of $x^* \text{ s-maj}_\rho\, x$.

$\boxed{k + 1}$

Let $\rho := \rho_1 \to \cdots \to \rho_{k+1} \to \tau$ and $\sigma := \rho_2 \to \cdots \to \rho_{k+1} \to \tau$, $\boldsymbol{r} = r_2, \ldots, r_{k+1}$ and $\boldsymbol{r^*} = r_2^*, \ldots, r_{k+1}^*$.

In order to simplify the reading of the following expressions, we will use

$$\begin{cases} A \\ B \end{cases} :\equiv A \wedge B$$

Notice that:

$$x^* \text{ s-maj}_\rho\, x \equiv \forall\, r_1, r_1^* \, (r_1^* \text{ s-maj}_\rho\, r_1 \to x^* r_1^* \text{ s-maj}_\sigma\, xr_1 \wedge x^* r_1^* \text{ s-maj}_\sigma\, x^* r_1) \tag{3.12}$$

$$\leftrightarrow \forall\, r_1, r_1^* \left(r_1^* \text{ s-maj}_\rho\, r_1 \to \begin{cases} \forall\, \boldsymbol{r}, \boldsymbol{r^*} \left(\bigwedge_{i=2}^{k+1} r_i^* \text{ s-maj}_{\rho_i}\, r_i \to \begin{cases} x^* r_1^* \boldsymbol{r^*} \text{ s-maj}_\tau\, xr_1 \boldsymbol{r} \\ x^* r_1^* \boldsymbol{r^*} \text{ s-maj}_\tau\, x^* r_1^* \boldsymbol{r} \end{cases} \right) \\ \forall\, \boldsymbol{r}, \boldsymbol{r^*} \left(\bigwedge_{i=2}^{k+1} r_i^* \text{ s-maj}_{\rho_i}\, r_i \to \begin{cases} x^* r_1^* \boldsymbol{r^*} \text{ s-maj}_\tau\, x^* r_1 \boldsymbol{r} \\ x^* r_1^* \boldsymbol{r^*} \text{ s-maj}_\tau\, x^* r_1^* \boldsymbol{r} \end{cases} \right) \end{cases} \right) \tag{3.13}$$

$$\leftrightarrow \forall\, r_1, r_1^*, \boldsymbol{r}, \boldsymbol{r^*} \left(r_1^* \text{ s-maj}_\rho\, r_1 \to \bigwedge_{i=2}^{k+1} r_i^* \text{ s-maj}_{\rho_i}\, r_i \to \begin{cases} x^* r_1^* \boldsymbol{r^*} \text{ s-maj}_\tau\, xr_1 \boldsymbol{r} \\ x^* r_1^* \boldsymbol{r^*} \text{ s-maj}_\tau\, x^* r_1 \boldsymbol{r} \\ x^* r_1^* \boldsymbol{r^*} \text{ s-maj}_\tau\, x^* r_1^* \boldsymbol{r} \end{cases} \right) \tag{3.14}$$

$$\leftrightarrow \forall\, r_1, r_1^*, \boldsymbol{r}, \boldsymbol{r^*} \left(\bigwedge_{i=1}^{k+1} r_i^* \text{ s-maj}_{\rho_i}\, r_i \to \begin{cases} x^* r_1^* \boldsymbol{r^*} \text{ s-maj}_\tau\, xr_1 \boldsymbol{r} \\ x^* r_1^* \boldsymbol{r^*} \text{ s-maj}_\tau\, x^* r_1 \boldsymbol{r} \\ x^* r_1^* \boldsymbol{r^*} \text{ s-maj}_\tau\, x^* r_1^* \boldsymbol{r} \end{cases} \right) \tag{3.15}$$

$$\leftrightarrow \forall\, r_1, r_1^*, \boldsymbol{r}, \boldsymbol{r^*} \left(\bigwedge_{i=1}^{k+1} r_i^* \text{ s-maj}_{\rho_i}\, r_i \to \begin{cases} x^* r_1^* \boldsymbol{r^*} \text{ s-maj}_\tau\, xr_1 \boldsymbol{r} \\ x^* r_1^* \boldsymbol{r^*} \text{ s-maj}_\tau\, x^* r_1 \boldsymbol{r} \end{cases} \right) \tag{3.16}$$

where (3.12) $\leftrightarrow$ (3.13) comes from the induction hypothesis, (3.13) $\leftrightarrow$ (3.14) is easily provable intuitionistically, (3.14) $\leftrightarrow$ (3.15) is a direct application of the importation and exportation rules (several times), and (3.15) $\rightarrow$ (3.16) is a simple weakening.

Finally, to prove the non-trivial assertion in (3.16) $\rightarrow$ (3.15), assume (3.16) and take any $s_1, s_1^*, \boldsymbol{s}, \boldsymbol{s^*}$ such that $\bigwedge_{i=1}^{k+1} s_i^* \text{ s-maj}_{\rho_i} s_i$. Now instantiate the quantified variables in (3.16) in the following way: $r_1 := s_1^*, r_1^* := s_1^*, \boldsymbol{r} := \boldsymbol{s}, \boldsymbol{r^*} := \boldsymbol{s^*}$, which is possible because by Lemma 3.3.2, $s_1^* \text{ s-maj}_{\rho_1} s_1^*$. This allows us to conclude $x^* s_1^* \boldsymbol{s^*} \text{ s-maj}_\tau x^* s_1^* \boldsymbol{s}$, as desired.

□

Lemma 3.4. WE-HA$^\omega$ proves the following:

1. $\max_\rho \text{ s-maj}_\rho \max_\rho$;
2. $x \text{ s-maj}_\rho x \wedge y \text{ s-maj}_\rho y \rightarrow \max_\rho xy \text{ s-maj}_\rho x$;
3. $x \text{ s-maj}_\rho x \wedge y \text{ s-maj}_\rho y \rightarrow \max_\rho xy \text{ s-maj}_\rho y$.

Proof of 1. By induction on the type:

$\boxed{0}$

We want to show:

$$\forall x, x^*, y, y^* \, (x^* \geq_0 x \wedge y^* \geq_0 y \rightarrow \max_0 x^* y^* \geq_0 \max_0 xy)$$

Notice that by Lemma 3.1.7 and Lemma 3.1.8, $\max_0 x^* y^* \geq_0 x^*$ and $\max_0 x^* y^* \geq_0 y^*$. As, by hypothesis, $x^* \geq_0 x$ and $y^* \geq_0 y$, the transitivity of $\geq_0$ gives us:

$$\max_0 x^* y^* \geq_0 x \wedge \max_0 x^* y^* \geq_0 y$$

and by Lemma 3.1.9, it follows that $\max_0 x^* y^* \geq_0 \max_0 xy$.

$\boxed{\rho \rightarrow \tau}$

By Lemma 3.3.5, it suffices to prove that for arbitrary x, x^*, y, y^*, z, z^* such that $x^* \text{ s-maj}_{\rho\rightarrow\tau} x$, $y^* \text{ s-maj}_{\rho\rightarrow\tau} y$ and $z^* \text{ s-maj}_\rho z$:

$$\max_{\rho\rightarrow\tau} x^* y^* z^* \text{ s-maj}_\tau \max_{\rho\rightarrow\tau} xyz$$

Now, by definition of $\max_{\rho\rightarrow\tau}$, we have $\max_{\rho\rightarrow\tau} x^* y^* z^* =_\tau \max_\tau (x^* z^*)(y^* z^*)$ and $\max_{\rho\rightarrow\tau} xyz =_\tau \max_\tau (xz)(yz)$, which taking into consideration:

- $x^* z^*$ s-maj xz because x^* s-maj x and z^* s-maj z
- $y^* z^*$ s-maj yz because y^* s-maj y and z^* s-maj z
- The induction hypothesis: $\max_\tau$ s-maj $\max_\tau$
- Lemma 3.3.1

yields the claim.

□

Proof of 2 and 3. We prove only 2, as the proof of 3 is very similar. By induction on the type:

$\boxed{0}$ Direct from Lemma 3.1.7.

$\boxed{\rho \rightarrow \tau}$

Chose arbitrary $x^{\rho\rightarrow\tau}, y^{\rho\rightarrow\tau}, r^\rho, (r^*)^\rho$ and assume:

(i) $x \text{ s-maj}_{\rho\rightarrow\tau} x$;

(ii) $y \text{ s-maj}_{\rho\rightarrow\tau} y$;

(iii) $r^* \text{ s-maj}_\rho r$

By Lemma 3.3.2 and (iii) we also know:

(iv) $r^* \text{ s-maj}_\rho r^*$

We need to show:

$$\max_{\rho\to\tau} xyr^* \text{ s-maj}_\tau\, xr \tag{3.17}$$
$$\max_{\rho\to\tau} xyr^* \text{ s-maj}_\tau\, \max_{\rho\to\tau} xyr \tag{3.18}$$

It is clear that (3.18) follows from 1, taking into consideration (i), (ii), (iii) and Lemma 3.3.5. As for (3.17), start by noticing that, by definition of $\max_{\rho\to\tau}$:

$$\max_{\rho\to\tau} xyr^* =_\tau \max_\tau (xr^*)(yr^*)$$

and:

(v) $xr^* \text{ s-maj}_\rho\, xr^*$ by (i) and (iv)

(vi) $yr^* \text{ s-maj}_\rho\, yr^*$ by (ii) and (iv)

It now follows by induction hypothesis that $\max_\tau(xr^*)(yr^*) \text{ s-maj}_\tau\, xr^*$, using (v) and (vi). Furthermore, by (i) and (iii), we know that $xr^* \text{ s-maj}_\tau\, xr$. So, finally, by transitivity of s-maj (Lemma 3.3.3), we are able to conclude (3.17).

□

The notion of strong majorizability is due to Marc Bezem. There is also an earlier notion of weak majoirizability due to William Howard that doesn't require the condition $x^*y^* \text{s-maj}_\tau x^*y$. The following theorems are also true when using weak majorizability, but the weaker version is not transitive (*i.e.*, we cannot prove an analogous statement to Lemma 3.3.3), and so we stick to strong majorizability for now. For similar statements to those mentioned in this section using weak majorizability, see [Kohlenbach, 2008].

Definition 3.5 (f^M)**.** For a term f of type $0 \to \rho$, we define f^M of type $0 \to \rho$ by type ρ induction, such that

$$f^M 0 =_\rho f0$$
$$f^M(Sn) =_\rho \max_\rho (f^M n)(f(Sn))$$

Remark 3.6. It is possible to define f^M using only type 0 induction. For details, see [Kohlenbach, 2008].

Lemma 3.7. If $\forall n^0\, xn \text{ s-maj}_\rho\, yn$, then for all m^0:

1. $x^M m \text{ s-maj}_\rho\, x^M m$
2. $x^M m \text{ s-maj}_\rho\, ym$
3. $x^M(Sm) \text{ s-maj}_\rho\, x^M m$

Proof. We start by stating our hypothesis:

$$\forall n^0\, xn \text{ s-maj } yn \tag{3.19}$$

Notice that by Lemma 3.3.2 and (3.19), we can add another hypothesis:

$$\forall n^0\, xn \text{ s-maj } xn \tag{3.20}$$

Now we prove each statement in turn:

1. $x^M m \text{ s-maj}_\rho\, x^M m$

By induction on m^0:

0 Taking into consideration that $x^M 0 = x0$, this follows by (3.20), instantiating $n := 0$.

$m \to Sm$

Notice that, by the definition of x^M and Lemma 3.3.1:

$$x^M(Sm) \text{ s-maj } x^M(Sm) \text{ iff } \max(x^M m)(x(Sm)) \text{ s-maj } \max(x^M m)(x(Sm))$$

and this follows by Lemma 3.4.1 and Lemma 3.3.5, noticing that:

- $x^M m$ s-maj $x^M m$ by induction hypothesis;
- $x(Sm)$ s-maj $x(Sm)$ by (3.20), instantiating $n := Sm$.

2. $x^M m$ s-maj$_\rho$ ym

By analyzing the cases $m =_0 0$ and $m =_0 Sm'$:

0 Taking into consideration that $x^M 0 = x0$, this follows by (3.19), instantiating $n := 0$.

Sm

Again by the definition of x^M and Lemma 3.3.1:

$$x^M(Sm) \text{ s-maj } y(Sm) \text{ iff } \max(x^M m)(x(Sm)) \text{ s-maj } y(Sm)$$

and by the transitivity of s-maj (Lemma 3.3.3), it suffices to prove:

$$\max(x^M m)(x(Sm)) \text{ s-maj } x(Sm) \tag{3.21}$$

$$x(Sm) \text{ s-maj } y(Sm) \tag{3.22}$$

Noticing that:

- $x^M m$ s-maj $x^M m$ by 1;
- $x(Sm)$ s-maj $x(Sm)$ by (3.20), instantiating $n := Sm$.

it is the case that (3.21) follows by Lemma 3.4.3.

As for (3.22), it is a direct consequence of (3.19), instantiating $n := Sm$;

3. $x^M(Sm)$ s-maj$_\rho$ $x^M m$

Yet again by the definition of x^M and Lemma 3.3.1:

$$x^M(Sm) \text{ s-maj } x^M m \text{ iff } \max(x^M m)(x(Sm)) \text{ s-maj } x^M m$$

and this is a direct consequence of Lemma 3.4.2, noticing simply that:

- $x^M m$ s-maj $x^M m$ by 1;
- $x(Sm)$ s-maj $x(Sm)$ by (3.20), instantiating $n := Sm$;

as had already been seen.

□

Lemma 3.8. WE-HA$^\omega \vdash \forall x^{0\to\rho}, y^{0\to\rho} \, (\forall n^0 \, (xn \text{ s-maj}_\rho\, yn) \to x^M \text{ s-maj}_{0\to\rho}\, y)$.

Proof. Choose arbitrary x, y of type $0 \to \rho$, such that

$$\forall n^0 \, (xn \text{ s-maj}_\rho\, yn) \tag{3.23}$$

We want to show:

$$\forall m, m^* \, (m^* \geq_0 m \to x^M m^* \text{ s-maj}_\rho\, ym \wedge x^M m^* \text{ s-maj}_\rho\, x^M m)$$

and we will do it by induction on $(m^*)^0$:

$m^* = 0$

In this case, as $m \leq m^* = 0$, we necessarily conclude $m = 0$. Furthermore, $x^M 0 = x0$. Then the result follows from (3.23), instantiating $n := 0$ and from Lemma 3.7.1.

$m^* \to Sm^*$

Taking into consideration that $m \leq Sm^* \leftrightarrow m < Sm^* \vee m = Sm^* \leftrightarrow m \leq m^* \vee m = Sm^*$, we consider both cases separately:

$\boxed{m \leq m^*}$ We want to show:

$$\forall m\,(m^* \geq m \to x^M(Sm^*) \text{ s-maj } ym \wedge x^M(Sm^*) \text{ s-maj } x^M m)$$

By transitivity of s-maj (Lemma 3.3.3), it suffices to prove:

$$x^M(Sm^*) \text{ s-maj } x^M m^* \tag{3.24}$$

$$\forall m\,(m^* \geq m \to x^M m^* \text{ s-maj } ym) \tag{3.25}$$

$$\forall m\,(m^* \geq m \to x^M m^* \text{ s-maj } x^M m) \tag{3.26}$$

Now (3.24) is a direct consequence of Lemma 3.7.3 and both (3.25) and (3.26) follow by induction hypothesis.

$\boxed{m = Sm^*}$ We need to show $x^M(Sm^*)$ s-maj $y(Sm^*)$, which is a direct consequence of Lemma 3.7.2, and $x^M(Sm^*)$ s-maj $x^M(Sm^*)$, which is a direct consequence of Lemma 3.7.1.

□

Theorem 3.9 (after Howard)**.** For every closed term t^ρ of WE-HA$^\omega$ it is possible to construct a closed term t^* with type ρ of WE-HA$^\omega$ such that:

$$\text{WE-HA}^\omega \vdash t^* \text{ s-maj}_\rho\, t$$

Proof. Induction on the structure of t.

$\boxed{0}$

Notice that:

$$\begin{aligned} 0 \text{ s-maj}_0\, 0 &\equiv 0 \geq_0 0 \\ &\equiv 0 >_0 0 \vee 0 =_0 0 \\ &\leftarrow 0 =_0 0 \end{aligned}$$

and so, by $0 =_0 0$, we conclude $0 \text{ s-maj}_0\, 0$.

$\boxed{S}$

Notice that:

$$\begin{aligned} S \text{ s-maj}_{0\to 0}\, S &\equiv \forall r^0, (r^*)^0\,(r^* \geq_0 r \to Sr^* \geq_0 Sr \wedge Sr^* \geq_0 Sr) \\ &\leftrightarrow \forall r^0, (r^*)^0\,(r^* \geq_0 r \to Sr^* \geq_0 Sr) \end{aligned}$$

which is a direct consequence of Lemma 3.1.5 and of the quantifier-free extensionality rule.

$\boxed{\Pi_{\rho,\tau}}$

Notice that:

$$\begin{aligned} &\Pi \text{ s-maj}_{\rho\to\tau\to\rho}\, \Pi \leftrightarrow \\ &\quad\leftrightarrow \forall r_1^\rho, (r_1^*)^\rho, r_2^\tau, (r_2^*)^\tau\,(r_1^* \text{ s-maj}_\rho\, r_1 \wedge r_2^* \text{ s-maj}_\tau\, r_2 \to \Pi r_1^* r_2^* \text{ s-maj}_\rho\, \Pi r_1 r_2 \wedge \Pi r_1^* r_2^* \text{ s-maj}_\rho\, \Pi r_1 r_2) \\ &\quad\leftrightarrow \forall r_1^\rho, (r_1^*)^\rho, r_2^\tau, (r_2^*)^\tau\,(r_1^* \text{ s-maj}_\rho\, r_1 \wedge r_2^* \text{ s-maj}_\tau\, r_2 \to r_1^* \text{ s-maj}_\rho\, r_1) \end{aligned}$$

where the first equivalence comes from Lemma 3.3.5 and the last equivalence uses the equalities $\Pi r_1^* r_2^* =_\rho r_1^*$ and $\Pi r_1 r_2 =_\rho r_1$, and Lemma 3.3.1.

$\boxed{\Sigma_{\delta,\rho,\tau}}$

Notice that:

$$\begin{aligned} &\Sigma \text{ s-maj}_{(\delta\to\rho\to\tau)\to(\delta\to\rho)\to\delta\to\tau}\, \Sigma \leftrightarrow \\ &\quad\leftrightarrow \forall r_1, r_1^*, r_2, r_2^*, r_3, r_3^*\,(r_1^* \text{ s-maj}_{\delta\to\rho\to\tau}\, r_1 \wedge r_2^* \text{ s-maj}_{\delta\to\rho}\, r_2 \wedge r_3^* \text{ s-maj}_\delta\, r_3 \to \\ &\qquad\qquad \to \Sigma r_1^* r_2^* r_3^* \text{ s-maj}_\tau\, \Sigma r_1 r_2 r_3 \wedge \Sigma r_1^* r_2^* r_3^* \text{ s-maj}_\tau\, \Sigma r_1 r_2 r_3) \\ &\quad\leftrightarrow \forall r_1, r_1^*, r_2, r_2^*, r_3, r_3^*\,(r_1^* \text{ s-maj}_{\delta\to\rho\to\tau}\, r_1 \wedge r_2^* \text{ s-maj}_{\delta\to\rho}\, r_2 \wedge r_3^* \text{ s-maj}_\delta\, r_3 \to \\ &\qquad\qquad \to r_1^* r_3^*(r_2^* r_3^*) \text{ s-maj}_\tau\, r_1 r_3(r_2 r_3)) \end{aligned}$$

where we have used again Lemma 3.3.5 and .1 for analogous purposes as in the proof of Π s-maj Π.

Let $r_1, r_1^*, r_2, r_2^*, r_3, r_3^*$ be any terms of the appropriate types, and assume r_i^* s-maj r_i for each $i \in \{1,2,3\}$. By the definition of r_2^* s-maj$_{\delta\to\rho}$ r_2 and the assumption r_3^* s-maj$_\delta$ r_3 we conclude

$$r_2^* r_3^* \text{ s-maj}_\rho\, r_2 r_3 \tag{3.27}$$

By Lemma 3.3.5, using r_1^* s-maj$_{\delta\to\rho\to\tau}$ r_1, r_3^* s-maj$_\delta$ r_3 and (3.27) we conclude

$$r_1^* r_3^* (r_2^* r_3^*) \text{ s-maj}_\tau\, r_1 r_3 (r_2 r_3)$$

which allows us to finally conclude Σ s-maj Σ.

$\boldsymbol{R_\rho}$

Suppose that $\boldsymbol{\rho} = \rho_1, \dots, \rho_k$, and let $\boldsymbol{y^*}, \boldsymbol{y}, \boldsymbol{z^*}, \boldsymbol{z}$ be of the types that ensure that the terms $\boldsymbol{R_\rho} x^0 \boldsymbol{yz}$ and $\boldsymbol{R_\rho} x^0 \boldsymbol{y^* z^*}$ are terms of WE-HA$^\omega$. Assume further that $\boldsymbol{y^*}$ s-maj $\boldsymbol{y}$ and $\boldsymbol{z^*}$ s-maj $\boldsymbol{z}$, where

$$\boldsymbol{y^*} \text{ s-maj } \boldsymbol{y} :\equiv \bigwedge_{i=1}^{k} y_i^* \text{ s-maj } y_i$$

We show by induction on x^0 that:

$$\forall x^0\, (\boldsymbol{R_\rho} x \boldsymbol{y^* z^*} \text{ s-maj } \boldsymbol{R_\rho} x \boldsymbol{yz}) \tag{3.28}$$

$x =_0 0$

In this case, by the definition of $\boldsymbol{R_\rho}$, $\boldsymbol{R_\rho} 0 \boldsymbol{y^* z^*} =_{\boldsymbol{\rho}} \boldsymbol{y^*}$ and $\boldsymbol{R_\rho} 0 \boldsymbol{yz} =_{\boldsymbol{\rho}} \boldsymbol{y}$. So the thesis follows by Lemma 3.3.1 and the hypothesis $\boldsymbol{y^*}$ s-maj $\boldsymbol{y}$.

$x \to Sx$

By the definition of $\boldsymbol{R_\rho}$:

$$\boldsymbol{R_\rho}(Sx)\boldsymbol{y^* z^*} =_{\boldsymbol{\rho}} \boldsymbol{z^*}(\boldsymbol{R_\rho} x \boldsymbol{y^* z^*})x$$
$$\boldsymbol{R_\rho}(Sx)\boldsymbol{yz} =_{\boldsymbol{\rho}} \boldsymbol{z}(\boldsymbol{R_\rho} x \boldsymbol{yz})x$$

Now clearly $x \geq_0 x$ and by induction hypothesis $\boldsymbol{R_\rho} x \boldsymbol{y^* z^*}$ s-maj$_{\boldsymbol{\rho}}$ $\boldsymbol{R_\rho} x \boldsymbol{yz}$. The result follows by Lemmas 3.3.5 and 3.3.1.

From (3.28) and Lemma 3.3.5, we conclude

$$\forall x^0\, (\boldsymbol{R_\rho} x \text{ s-maj}_{\boldsymbol{\rho}}\, \boldsymbol{R_\rho} x)$$

and Lemma 3.8 gives us the final result:

$$(R_i)_{\boldsymbol{\rho}}^M \text{ s-maj}_{\rho_i}\, (R_i)_{\boldsymbol{\rho}} \quad i \in \{1, \dots, k\}$$

The result now follows from the fact that if t^* s-maj t and u^* s-maj u, then $t^* u^*$ s-maj tu. □

Corollary 3.10. Projection terms of WE-HA$^\omega$, *i.e.*, terms of the form

$$\lambda\, x_1, \dots, x_k \,.\, x_i$$

strongly majorize themselves.

Proof. This is direct from the proof of Theorem 3.9, remembering that projection terms are nothing more than a combination of several Π and Σ (cf. Definition 1.15). □

3.2 Monotone Functional Interpretation

For many applications of Gödel's interpretation, the exact witness terms $\boldsymbol{T}$ are not important. What really matters is to know that such terms exist, and some bound over them. In other words, for each formula $A(\boldsymbol{a})$ it would suffice to know closed terms $\boldsymbol{T^*}$ such that:

$$\exists \boldsymbol{x}\,(\boldsymbol{T^*} \text{ s-maj } \boldsymbol{x} \wedge \forall \boldsymbol{a}, \boldsymbol{y}\, A_D(\boldsymbol{xa}, \boldsymbol{y}, \boldsymbol{a}))$$

If that in this case, we say that $\boldsymbol{T^*}$ satisfies the monotone functional interpretation of A.

We now define a set of closed formulas that will have a trivial monotone interpretation (similar to the set $\mathcal{P}$ in the case of the 'dialectica' interpretation). Let Δ be a set of formulas of the form:

$$\forall \boldsymbol{a^\delta}\, \exists \boldsymbol{b^\sigma}\,(\boldsymbol{b} \leq_{\boldsymbol{\sigma}} \boldsymbol{ra} \wedge \forall \boldsymbol{c^\gamma}\, A_0(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}))$$

where A_0 is a quantifier-free formula with no free variables except the ones in $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}$ and $\boldsymbol{r}$ is a tuple of closed terms of suitable types of WE-HA$^\omega$. We also used for the first time the notation:

$$\boldsymbol{t} \leq_{\boldsymbol{\rho}} \boldsymbol{u} :\equiv \bigwedge_{i=1}^{k} (t_i \leq_{\rho_i} u_i)$$

Furthermore, given a set Δ, we define the corresponding set of Skolem normal forms:

$$\widetilde{\Delta} := \{\widetilde{\varphi} :\equiv \exists \boldsymbol{B}\,(\boldsymbol{B} \leq \boldsymbol{r} \wedge \forall \boldsymbol{a}, \boldsymbol{c}\, A_0(\boldsymbol{a}, \boldsymbol{Ba}, \boldsymbol{c}))\ :\ \varphi :\equiv \forall \boldsymbol{a^\delta}\, \exists \boldsymbol{b^\sigma}\,(\boldsymbol{b} \leq_{\boldsymbol{\sigma}} \boldsymbol{ra} \wedge \forall \boldsymbol{c^\gamma}\, A_0(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c})) \in \Delta\}$$

Lemma 3.11. WE-HA$^\omega$ + AC$^\omega$ $\vdash \varphi \to \widetilde{\varphi}$.

Proof. One first shows that:

$$\text{b-AC}^{\delta,\rho} : \forall Z^{\delta\to\rho}\,(\forall x^\delta\, \exists y^\rho\,(y \leq_\rho Zx \wedge A(x, y, Z)) \to \exists Y^{\delta\to\rho}\,(Y \leq_{\delta\to\rho} Z \wedge \forall x\, A(x, Yx, Z)))$$

is a consequence of AC$^\omega$ for all types δ, ρ, and then the result follows easily. ☐

Theorem 3.12 (Soundness for the monotone functional interpretation)**.** Let Δ be as defined above, and $A(\boldsymbol{a}) \in \mathcal{L}(\text{WE-HA}^\omega)$ containing only $\boldsymbol{a}$ free. Then:

$$\text{WE-HA}^\omega + \text{AC}^\omega + \text{IP}^\omega_\forall + \text{M}^\omega + \Delta \vdash A(\boldsymbol{a})$$

implies

$$\text{WE-HA}^\omega + \widetilde{\Delta} \vdash \exists \boldsymbol{x}\,(\boldsymbol{T^*} \text{ s-maj } \boldsymbol{x} \wedge \forall \boldsymbol{a}, \boldsymbol{y}\, A_D(\boldsymbol{xa}, \boldsymbol{y}, \boldsymbol{a}))$$

where $\boldsymbol{T^*}$ is a tuple of closed terms of WE-HA$^\omega$ which can be extracted from a given proof of $A(\boldsymbol{a})$.

Proof. The proof is by induction on the length of the proof of $A(\boldsymbol{a})$. For the axioms (excluding Δ), the result follows from the soundness of the 'dialectica' interpretation (Theorem 2.7) and Howard's theorem (Theorem 3.9). However, the construction of $\boldsymbol{T^*}$ is often simpler than the construction of $\boldsymbol{T}$, which is the case for the axiom $A \to A \wedge A$, for example. We do not give every step of the proof in detail, but mention only some cases:

$\boxed{A \to A \wedge A}$

Recall Gödel's translation of this axiom:

$$[A \to A \wedge A]^D \equiv \exists \boldsymbol{U}, \boldsymbol{Q}, \boldsymbol{Y}\ \forall \boldsymbol{x}, \boldsymbol{v}, \boldsymbol{r}\,(A_D(\boldsymbol{x}, \boldsymbol{Yxvr}, \boldsymbol{a}) \to A_D(\boldsymbol{Ux}, \boldsymbol{v}, \boldsymbol{a}) \wedge A_D(\boldsymbol{Qx}, \boldsymbol{r}, \boldsymbol{a}))$$

and the terms found during the proof of Theorem 2.7:

$$\begin{aligned} \boldsymbol{T_U} &:= \lambda\, \boldsymbol{a}, \boldsymbol{x}\,.\, \boldsymbol{x} \\ \boldsymbol{T_Q} &:= \lambda\, \boldsymbol{a}, \boldsymbol{x}\,.\, \boldsymbol{x} \\ \boldsymbol{T_Y} &:= \lambda\, \boldsymbol{a}, \boldsymbol{x}, \boldsymbol{v}, \boldsymbol{r}\,.\, \begin{cases} \boldsymbol{v} & \text{if } \neg A_D(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{a}) \\ \boldsymbol{r} & \text{if } A_D(\boldsymbol{x}, \boldsymbol{v}, \boldsymbol{a}) \end{cases} \end{aligned}$$

By Corollary 3.10, $\boldsymbol{T_U}$ and $\boldsymbol{T_Q}$ strongly majorize themselves.

Consider now $\boldsymbol{T^*_Y} := \lambda\, \boldsymbol{a}, \boldsymbol{x}, \boldsymbol{v}, \boldsymbol{r}\,.\, \max \boldsymbol{v}\, \boldsymbol{r}$. It is the case that $\boldsymbol{T^*_Y}$ s-maj $\boldsymbol{T_Y}$.

Induction schema

We use again the equivalent induction rule:

$$\frac{A(0,\boldsymbol{a'}), A(z,\boldsymbol{a'}) \to A(Sz,\boldsymbol{a'})}{A(z,\boldsymbol{a'})}$$

Notice that:

$$\begin{aligned} [A(0,\boldsymbol{a'})]^D &\equiv \exists \boldsymbol{x}\, \forall \boldsymbol{y}\, A_D(\boldsymbol{x},\boldsymbol{y},0,\boldsymbol{a'}) \\ [A(z,\boldsymbol{a'}) \to A(Sz,\boldsymbol{a'})]^D &\equiv \exists \boldsymbol{U},\boldsymbol{Y}\, \forall \boldsymbol{x},\boldsymbol{v}\, (A_D(\boldsymbol{x},\boldsymbol{Yxv},z,\boldsymbol{a'}) \to A_D(\boldsymbol{Ux},\boldsymbol{v},Sz,\boldsymbol{a'})) \\ [A(z,\boldsymbol{a'})]^D &\equiv \exists \boldsymbol{x}\, \forall \boldsymbol{y}\, A_D(\boldsymbol{x},\boldsymbol{y},z,\boldsymbol{a'}) \end{aligned}$$

By induction hypothesis, there are closed terms $\boldsymbol{T_1^*}, \boldsymbol{T_2^*}$ and $\boldsymbol{T_3^*}$ such that:

$\exists \boldsymbol{X}\, (\boldsymbol{T_1^*} \text{ s-maj } \boldsymbol{X} \wedge \forall \boldsymbol{a'},\boldsymbol{y}\, A_D(\boldsymbol{Xa'},\boldsymbol{y},0,\boldsymbol{a'}))$
$\exists \boldsymbol{\mathcal{U}},\boldsymbol{\mathcal{Y}}\, (\boldsymbol{T_2^*} \text{ s-maj } \boldsymbol{\mathcal{U}} \wedge \boldsymbol{T_3^*} \text{ s-maj } \boldsymbol{\mathcal{Y}} \wedge \forall z,\boldsymbol{a'},\boldsymbol{x},\boldsymbol{v}\, (A_D(\boldsymbol{x},\boldsymbol{\mathcal{Y}}z\boldsymbol{a'xv},z,\boldsymbol{a'}) \to A_D(\boldsymbol{\mathcal{U}}z\boldsymbol{a'x},\boldsymbol{v},Sz,\boldsymbol{a'})))$

Define $\boldsymbol{T}_4$ by recursion such that:

$$\begin{cases} \boldsymbol{T}_4 0\boldsymbol{a'} = \boldsymbol{T_1^*a'} \\ \boldsymbol{T}_4(Sz)\boldsymbol{a'} = \boldsymbol{T_2^*}z\boldsymbol{a'}(\boldsymbol{T}_4 z\boldsymbol{a'}) \end{cases}$$

and $\boldsymbol{T_4^*} := \boldsymbol{T}_4^M$.
Now it is easy to verify that $\boldsymbol{T_4^*}$ s-maj $\boldsymbol{w}$, where $\boldsymbol{w}$ is defined by induction as:

$$\begin{cases} \boldsymbol{w}0\boldsymbol{a'} = \boldsymbol{Xa'} \\ \boldsymbol{w}(Sz)\boldsymbol{a'} = \boldsymbol{\mathcal{U}}z\boldsymbol{a'}(\boldsymbol{w}z\boldsymbol{a'}) \end{cases}$$

and, in the same way as it did in the soundness proof of the 'dialectica' interpretation (Theorem 2.7), $\boldsymbol{w}$ is such that:

$$\forall \boldsymbol{y}\, A_D(\boldsymbol{w}y\boldsymbol{a'},\boldsymbol{y},z,\boldsymbol{a'})$$

Δ

Take an axiom in Δ:

$$\varphi \equiv \forall \boldsymbol{a^\delta}\, \exists \boldsymbol{b^\sigma}\, (\boldsymbol{b} \leq_{\boldsymbol{\sigma}} \boldsymbol{ra} \wedge \forall \boldsymbol{c^\gamma}\, A_0(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}))$$

Noting that, because both $\boldsymbol{b} \leq_{\boldsymbol{\sigma}} \boldsymbol{ra} \equiv \forall \boldsymbol{v}\, (\boldsymbol{bv} \leq_0 \boldsymbol{rav})$ and $\forall \boldsymbol{c}\, A_0(\boldsymbol{a},\boldsymbol{b},\boldsymbol{c})$ are purely universal formulas, their 'dialectica' interpretation doesn't change them by Remark 2.2.3, we have:

$$\begin{aligned} \varphi^D &\equiv \exists \boldsymbol{B}\, \forall \boldsymbol{a},\boldsymbol{v},\boldsymbol{c}\, (\boldsymbol{Bav} \leq_{\boldsymbol{0}} \boldsymbol{rav} \wedge A_0(\boldsymbol{a},\boldsymbol{Ba},\boldsymbol{c})) \\ &\leftrightarrow \exists \boldsymbol{B}\, (\boldsymbol{B} \leq \boldsymbol{r} \wedge \forall \boldsymbol{a},\boldsymbol{c}\, A_0(\boldsymbol{a},\boldsymbol{Ba},\boldsymbol{c})) \end{aligned}$$

where the equivalence comes from three simple remarks:

- $\forall$ commutes with $\wedge$;
- if $x \notin \text{fv}(t)$, $\forall x\, t \leftrightarrow t$;
- $\boldsymbol{B} \leq \boldsymbol{r} \leftrightarrow \forall \boldsymbol{a},\boldsymbol{v}\, (\boldsymbol{Bav} \leq_{\boldsymbol{0}} \boldsymbol{rav})$.

We want to find a tuple of closed terms $\boldsymbol{T^*}$ such that:

(i) $\boldsymbol{T^*}$ s-maj $\boldsymbol{B}$;

(ii) $\boldsymbol{B} \leq \boldsymbol{r} \wedge \forall \boldsymbol{a},\boldsymbol{c}\, A_0(\boldsymbol{a},\boldsymbol{Ba},\boldsymbol{c})$.

Choose $\boldsymbol{T^*}$ as a tuple of terms that strongly majorize $\boldsymbol{r}$ (which exists by Theorem 3.9). Then by $\boldsymbol{T^*}$ s-maj $\boldsymbol{r}$, $\boldsymbol{r} \geq \boldsymbol{B}$ and Lemma 3.3.4 we obtain (i). As for, (ii), it follows from $\widetilde{\Delta}$.

□

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