Admissibility and Unification: Relativised

Mojtaba Mojtahedi

Joint work with: Konstantinos Papafilippou

Ghent University

June 4, 2025

Unification

- Given A, unification asks for all substitutions θ s.t.

$$\vdash \theta(A)$$

- If θ unifies A then $\lambda\theta$ also unifies it.
- We say θ is more general than γ if there is some λ s.t. $\gamma = \lambda\theta$.
- Complete set of unifiers: a set of unifiers that every unifier is a subunifier of it.

Admissible Rules

Intuitively, $\dfrac{A}{B}$ is admissible (also written as $A \mathrel{|\!\sim} B$) if its addition to the logic would not cause any additional theorem in the logic.

Admissible Rules: precise definition

$$A \mathrel{|\!\sim} B \quad \text{iff} \quad \forall \theta \, (\vdash \theta(A) \; \Rightarrow \vdash \theta(B)).$$

Admissible Rules: example

Example.

$x \wedge (x \to y) \mathrel{|\!\sim} y$. This usually simplified as

$$\frac{A \qquad A \to B}{B}$$

In this notation the arbitrary substitution θ which $\theta(x) = A$ and $\theta(y) = B$ is implicit.

Admissible Rules: Classical Logic

Observation.

In classical logic, $A \mathrel{|\!\sim} B$ iff $\vdash A \to B$.

Proof.

$\vdash A \to B$ implies $A \mathrel{|\!\sim} B$: Obvious.
Let $\nvdash A \to B$. Then there is a truth-falsity substitution θ such that $\vdash \theta(A) \leftrightarrow \top$ and $\nvdash \theta(B) \leftrightarrow \bot$. Thus $\vdash \theta(A)$ and $\nvdash \theta(B)$. ❑

Admissibile rules: Intuitionistic Logic

Harrop 1960

$\neg x \to (y \vee z) \mathrel{|\!\sim} (\neg x \to y) \vee (\neg x \to z).$

Proof.

By Kripke semantics. □

Theorem

If A has a finite complete set of unifiers, then admissibility of $A \mid\!\sim B$ is decidable for every B.

Proof.

Let Θ be a finite complete set of unifiers of A. Then for $A \mid\!\sim B$, it is enough to check that for every $\theta \in \Theta$ we have $\vdash \theta(B)$, which is decidable, assuming the decidability of $\vdash$. ❑

Given A, we say that θ is A-projection if for every variable x

$$A \vdash \theta(x) \leftrightarrow x.$$

Observation.

A-projections are more general than all unifiers of A.

Proof. Let γ unifies A. Then $\gamma(A) \vdash \gamma\theta(x) \leftrightarrow \gamma(x)$ and thus $\vdash \gamma\theta(x) \leftrightarrow \gamma(x)$ for every variable x.

Definition.

A is called projective iff there is an A-projection unifier.

Observation:

Every unifiable formula has a one-element complete set of unifiers.

Proof. Let $\vdash \theta(A)$.

- $\epsilon_\theta(x) := (A \wedge x) \vee (\neg A \wedge \theta(x))$.
- ϵ_θ is A-projection.
- $A \vdash \epsilon_\theta(A) \leftrightarrow A$ and then $A \vdash \epsilon_\theta(A)$.
- $\neg A \vdash \epsilon_\theta(x) \leftrightarrow \theta(x)$ then $\neg A \vdash \epsilon_\theta(A) \leftrightarrow \theta(A)$.
- $\neg A \vdash \epsilon_\theta(A)$.
- $\vdash \epsilon_\theta(A)$.

$x \vee \neg x$ does not have a most general unifier. All unifiers of it are $\theta(x) := \top$ and $\theta(x) := \bot$.

Theorem (S. Ghilardi 1999)

The unification type of Intuitionistic Logic is finitary, i.e. for every formula there is a finite complete set of unifiers.

Application (R. Iemhoff 2001)

Completeness of a base for admissible rules of Intuitionistic Logic.

Base for admissible rules of IPC (Iemhoff 2001)

- $A \rhd B$ whenever $\mathsf{IPC} \vdash A \to B$.
- (Visser) For $B = \bigwedge_{i=1}^{n} E_i \to F_i$ and $C = \bigvee_{i=n+1}^{n+m} E_i$:

$$(B \to C) \rhd \bigvee_{i=1}^{n+m} B \xrightarrow{\perp} E_i$$

$$\frac{A \rhd B \qquad B \rhd C}{A \rhd C}\ \text{Cut} \qquad\qquad \frac{A \rhd B \qquad A \rhd C}{A \rhd (B \wedge C)}\ \text{Conj}$$

$$\frac{A \rhd C \qquad B \rhd C}{(A \vee B) \rhd C}\ \text{Disj}$$

Intuitionistic Logic (Continued . . .)

Admissibile Rules for Mathematical Theories?

$$A \mathrel{|\!\sim}_{\mathsf{T}} B \text{ iff } \forall\alpha \text{ ``}\mathsf{T} \vdash \alpha(A) \text{ implies } \mathsf{T} \vdash \alpha(B)\text{''}.$$

Special case (D. de Jongh 1970)

$\top \mathrel{|\!\sim}_{\mathsf{HA}} A$ iff $\mathsf{IPC} \vdash A$.

Admissibile Rules for Mathematical Theories?

$$A \mathrel{|\!\sim}_{\mathsf{T}} B \text{ iff } \forall\alpha \text{ ``}\mathsf{T} \vdash \alpha(A) \text{ implies } \mathsf{T} \vdash \alpha(B)\text{''}.$$

Special case (D. de Jongh 1970)

$\top \mathrel{|\!\sim}_{\mathsf{HA}} A$ iff $\mathsf{IPC} \vdash A$.

Application (A. Visser 2002)

Propositional admissible rules of HA are equal to admissible rules of IPC.

Intuitionistic Logic (Continued . . .)

Admissibile Rules for Mathematical Theories?

$$A \mathrel{|\!\!\sim}_{\mathsf{T}} B \text{ iff } \forall\alpha \text{ ``}\mathsf{T} \vdash \alpha(A) \text{ implies } \mathsf{T} \vdash \alpha(B)\text{''}.$$

Special case (D. de Jongh 1970)

$\top \mathrel{|\!\!\sim}_{\mathsf{HA}} A$ iff $\mathsf{IPC} \vdash A$.

Application (A. Visser 2002)

Propositional admissible rules of HA are equal to admissible rules of IPC.

Special case (R. Pasman 2022)

Propositional admissible rules of Constructive Set Theory (CZF) are equal to admissible rules of IPC.

Extending language by parameters

- We assume that the language also has a set of atomic constants (parameters).
- x, y for variables and p, q for parameters.
- Substitutions leave parametrs unchanged.
- In CL: Every unifiable formula is projective.
- In IL: Every unifiable formula has a finite complete set of unifiers.

- $A := p \wedge x$ can not be projective, since it is not unifiable.
- Instead of unifiers, we look for E-fiers for some parametric (variable-free) formula E.
- An E-fier of A is a substitution θ s.t. $\vdash \theta(A) \leftrightarrow E$.
- We say that A is par-projective, if there is some parametric E and A-projection E-fier for A:

$$\vdash \theta(A) \leftrightarrow E \quad \text{and} \quad A \vdash \theta(x) \leftrightarrow x.$$

In this case E is called a par-projection of A.

Observation.

Every par-projective formula has a unique par-projection.

Proof. For $i \in \{1, 2\}$ let θ_i be an A-projection E_i-fier of A.

Observation.

Every par-projective formula has a unique par-projection.

Proof. For $i \in \{1, 2\}$ let θ_i be an A-projection E_i-fier of A.

- $A \vdash \theta_1(A) \leftrightarrow A$. (by A-prjectiveness)

Observation.

Every par-projective formula has a unique par-projection.

Proof. For $i \in \{1, 2\}$ let θ_i be an A-projection E_i-fier of A.

- $A \vdash \theta_1(A) \leftrightarrow A$. (by A-prjectiveness)
- $\vdash A \to E_1$.

Observation.

Every par-projective formula has a unique par-projection.

Proof. For $i \in \{1, 2\}$ let θ_i be an A-projection E_i-fier of A.

- $A \vdash \theta_1(A) \leftrightarrow A$. (by A-prjectiveness)
- $\vdash A \to E_1$.
- $\vdash \theta_2(A \to E_1)$.

Observation.

Every par-projective formula has a unique par-projection.

Proof. For $i \in \{1, 2\}$ let θ_i be an A-projection E_i-fier of A.

- $A \vdash \theta_1(A) \leftrightarrow A$. (by A-prjectiveness)
- $\vdash A \to E_1$.
- $\vdash \theta_2(A \to E_1)$.
- $\vdash \theta_2(A) \to E_1$.

Observation.

Every par-projective formula has a unique par-projection.

Proof. For $i \in \{1,2\}$ let θ_i be an A-projection E_i-fier of A.

- $A \vdash \theta_1(A) \leftrightarrow A$. (by A-prjectiveness)
- $\vdash A \to E_1$.
- $\vdash \theta_2(A \to E_1)$.
- $\vdash \theta_2(A) \to E_1$.
- $\vdash E_2 \to E_1$.

Observation.

Every par-projective formula has a unique par-projection.

Proof. For $i \in \{1, 2\}$ let θ_i be an A-projection E_i-fier of A.

- $A \vdash \theta_1(A) \leftrightarrow A$. (by A-prjectiveness)
- $\vdash A \to E_1$.
- $\vdash \theta_2(A \to E_1)$.
- $\vdash \theta_2(A) \to E_1$.
- $\vdash E_2 \to E_1$.
- Similarly $\vdash E_1 \to E_2$.

Connection to UPI

Given A, the Uniform Post-Interpolant of A with respect to par is defined as a formula A^{par} s.t.:

1. A^{par} is parametric,
2. $\vdash A \to A^{\mathsf{par}}$,
3. For every parametric E with $\vdash A \to E$, we have $\vdash A^{\mathsf{par}} \to E$.

It is well-known that CL and IL both have UI.

Observation.

The unique par-projection of A is A^{par}.

Proof. Let θ be an A-projection E-fier of A.

Connection to UPI

Given A, the Uniform Post-Interpolant of A with respect to par is defined as a formula A^{par} s.t.:

1. A^{par} is parametric,
2. $\vdash A \to A^{\mathsf{par}}$,
3. For every parametric E with $\vdash A \to E$, we have $\vdash A^{\mathsf{par}} \to E$.

It is well-known that CL and IL both have UI.

Observation.

The unique par-projection of A is A^{par}.

Proof. Let θ be an A-projection E-fier of A.

- $A \vdash \theta(A) \leftrightarrow A$. (by A-projectiveness)

Connection to UPI

Given A, the Uniform Post-Interpolant of A with respect to par is defined as a formula A^{par} s.t.:

1. A^{par} is parametric,
2. $\vdash A \to A^{\mathsf{par}}$,
3. For every parametric E with $\vdash A \to E$, we have $\vdash A^{\mathsf{par}} \to E$.

It is well-known that CL and IL both have UI.

Observation.

The unique par-projection of A is A^{par}.

Proof. Let θ be an A-projection E-fier of A.

- $A \vdash \theta(A) \leftrightarrow A$. (by A-projectiveness)
- $\vdash A \to E$ and E is parametric.

Connection to UPI

Given A, the Uniform Post-Interpolant of A with respect to par is defined as a formula A^{par} s.t.:

1. A^{par} is parametric,
2. $\vdash A \to A^{\mathsf{par}}$,
3. For every parametric E with $\vdash A \to E$, we have $\vdash A^{\mathsf{par}} \to E$.

It is well-known that CL and IL both have UI.

Observation.

The unique par-projection of A is A^{par}.

Proof. Let θ be an A-projection E-fier of A.

- $A \vdash \theta(A) \leftrightarrow A$. (by A-projectiveness)
- $\vdash A \to E$ and E is parametric.
- Take parametric F s.t. $\vdash A \to F$.

Connection to UPI

Given A, the Uniform Post-Interpolant of A with respect to par is defined as a formula A^{par} s.t.:

1. A^{par} is parametric,
2. $\vdash A \to A^{\mathsf{par}}$,
3. For every parametric E with $\vdash A \to E$, we have $\vdash A^{\mathsf{par}} \to E$.

It is well-known that CL and IL both have UI.

Observation.

The unique par-projection of A is A^{par}.

Proof. Let θ be an A-projection E-fier of A.

- $A \vdash \theta(A) \leftrightarrow A$. (by A-projectiveness)
- $\vdash A \to E$ and E is parametric.
- Take parametric F s.t. $\vdash A \to F$.
- $\vdash \theta(A) \to F$.

Connection to UPI

Given A, the Uniform Post-Interpolant of A with respect to par is defined as a formula A^{par} s.t.:

1. A^{par} is parametric,
2. $\vdash A \to A^{\mathsf{par}}$,
3. For every parametric E with $\vdash A \to E$, we have $\vdash A^{\mathsf{par}} \to E$.

It is well-known that CL and IL both have UI.

Observation.

The unique par-projection of A is A^{par}.

Proof. Let θ be an A-projection E-fier of A.

- $A \vdash \theta(A) \leftrightarrow A$. (by A-projectiveness)
- $\vdash A \to E$ and E is parametric.
- Take parametric F s.t. $\vdash A \to F$.
- $\vdash \theta(A) \to F$.
- Thus $\vdash E \to F$.

par-projectivity for CL

Theorem (Papafilippou & M.)

Every formula is par*-projective.*

Projectivity for IL

- Kripke models for IL: finite and rooted.

Projectivity for IL

- Kripke models for IL: finite and rooted.
- $\mathcal{K}$ is a variant of $\mathcal{K}'$ iff they share the same frame, and they have the same valuations except at the root.

Projectivity for IL

- Kripke models for IL: finite and rooted.
- $\mathcal{K}$ is a variant of $\mathcal{K}'$ iff they share the same frame, and they have the same valuations except at the root.
- $\mathcal{K} \Vdash^{-} A$ iff for every w other than the root $\mathcal{K}, w \Vdash A$.

Projectivity for IL

- Kripke models for IL: finite and rooted.
- $\mathcal{K}$ is a variant of $\mathcal{K}'$ iff they share the same frame, and they have the same valuations except at the root.
- $\mathcal{K} \Vdash^{-} A$ iff for every w other than the root $\mathcal{K}, w \Vdash A$.
- A is extendable if every $\mathcal{K} \Vdash^{-} A$ has a variant $\mathcal{K}' \Vdash A$.

Projectivity for IL

- Kripke models for IL: finite and rooted.
- $\mathcal{K}$ is a variant of $\mathcal{K}'$ iff they share the same frame, and they have the same valuations except at the root.
- $\mathcal{K} \Vdash^- A$ iff for every w other than the root $\mathcal{K}, w \Vdash A$.
- A is extendable if every $\mathcal{K} \Vdash^- A$ has a variant $\mathcal{K}' \Vdash A$.
- I.e. : A is extendable if every finite set of Kripke models of A can be extended from below s.t. it also be a model of A.

Theorem (S. Ghilardi 1999)

A formula is projective iff it is extendable.

par-projectivity for IL

We say that $\mathcal{K}'$ is a par-variant of $\mathcal{K}$ if they share

1. same frame,
2. same valuation for par,
3. same valuation for variables at any world except the root.

We say that A is E-extendable if

- $\vdash A \to E$,
- Every $\mathcal{K} \Vdash^{-} A$ with $\mathcal{K} \Vdash E$ has a par-variant $\mathcal{K}' \Vdash A$.

Theorem (Papafilippou & M.)

A formula is E-projective iff it is E-extendable.

Connection to standard projectivity

Question

Can we express par-projectivity through standard projectivity?

Connection to standard projectivity

Question

Can we express par-projectivity through standard projectivity?

Theorem (Papafilippou & M.)

par-*projectivity is equivalent to projectivity of* $A^{\mathsf{par}} \to A$.

Proof.

Right-to-Left: Take some $(A^{\mathsf{par}} \to A)$-projection θ that unifies $A^{\mathsf{par}} \to A$. The same θ is also A-projection and A^{par}-fier.

Right-to-Left: Not straightforward. We could prove it separately for CL and IL. For the case of IL we had to use par-extendibility.

Why interesting?

1. It is a natural generalization of an important tool.

Why interesting?

1. It is a natural generalization of an important tool.
2. Decidability of Admissible Rules of extensions of intuitionistic logic by parametric axioms.

Why interesting?

1. It is a natural generalization of an important tool.
2. Decidability of Admissible Rules of extensions of intuitionistic logic by parametric axioms.
3. The following application:

Theorem (Papafilippou & M.)

The unification type of parametric extensions of IL are finitary.

Why interesting?

1. It is a natural generalization of an important tool.
2. Decidability of Admissible Rules of extensions of intuitionistic logic by parametric axioms.
3. The following application:

Theorem (Papafilippou & M.)

The unification type of parametric extensions of IL are finitary.

4. It showed up naturally during my long journey for the problem of Intuitionistic Provability Logic.

Digression: Intuitionistic Provability Logic

- Problem: Complete axiomatization and decidability of Provability Logic of HA.
- This question was taken up by A. Visser and D. de Jongh and their students since late 70.
- A. Visser 1981: decidability of leterless fragment.
- M. Ardeshir & M. 2018: The Σ-provability logic of HA.
- M. 2022: characterization and decidability of intuitionistic provability logic.

Relative Admissibility

In the same manner that admissibility relies on standard unification problem, we have relative admissibility, best fit for parametric unification.

Relative Admissibility

In the same manner that admissibility relies on standard unification problem, we have relative admissibility, best fit for parametric unification.

Definition.

$$A \mathrel{|\!\!\sim}_E B \quad \text{iff} \quad \forall\theta\ (\vdash \theta(E \to A) \ \Rightarrow \vdash \theta(E \to B))$$

Relative Admissibility

In the same manner that admissibility relies on standard unification problem, we have relative admissibility, best fit for parametric unification.

Definition.

$$A \mathrel{|\!\!\sim}_E B \quad \text{iff} \quad \forall\theta \; (\vdash \theta(E \to A) \; \Rightarrow \vdash \theta(E \to B))$$

This definition is just standard admissibility for the logic extended by E.

Relative Admissibility

In the same manner that admissibility relies on standard unification problem, we have relative admissibility, best fit for parametric unification.

Definition.

$$A \mathrel{|\!\sim}_E B \quad \text{iff} \quad \forall\theta\ (\vdash \theta(E \to A) \ \Rightarrow \vdash \theta(E \to B))$$

This definition is just standard admissibility for the logic extended by E.

Definition.

$$\mathrel{|\!\sim}_\Gamma := \bigcap_{E \in \Gamma \cap \mathcal{L}(\mathsf{par})} \mathrel{|\!\sim}_E \quad \text{or equivalently:}$$

$$A \mathrel{|\!\sim}_\Gamma B \quad \text{iff} \quad \forall E \in \Gamma \cap \mathcal{L}(\mathsf{par})\ A \mathrel{|\!\sim}_E B$$

Relative Admissibility (continued)

Theorem (Papafilippou & M.)

For every Γ *closed under parameter-substitutions,* $\mid\!\sim_\Gamma$ *is equal to* $\vdash$.

Admissibility relative to NNIL

Theorem (M. 2022)

$\mathrel{|\!\sim}_{\mathsf{NNIL}}$ *is decidable.*

Mojtahedi, Mojtaba. "Relative Unification in Intuitionistic Logic: Towards provability logic of HA." (arXiv 2022).

Mojtahedi, Mojtaba. "On Provability Logic of HA." (arXiv 2022).

NNIL-formulas

NNIL is the set of propositional formulas with No Nested Implications to the Left:

- $a \in \mathsf{NNIL}$ for atomic a.
- NNIL is closed under conjunctions and disjunctions.
- $A \in \mathsf{NNIL}$ implies $a \to A \in \mathsf{NNIL}$ for every atomic a.

This class of formula is due to Albert Visser and plays crucial role in several aspects related to Intuitionistic Logic.

Axiomatization of $\mathrel{|\!\!\sim}_{\mathsf{NNIL}}$ (M. 2022)

- $A \rhd B$ whenever $\mathsf{IPC} \vdash A \to B$.
- (Visser) For $B = \bigwedge_{i=1}^{n} E_i \to F_i$ and $C = \bigvee_{i=n+1}^{n+m} E_i$:

$$(B \to C) \rhd \bigvee_{i=1}^{n+m} B \xrightarrow{\mathsf{par}} E_i$$

$$\frac{A \rhd B \qquad B \rhd C}{A \rhd C}\ \text{Cut} \qquad\qquad \frac{A \rhd B \qquad A \rhd C}{A \rhd (B \wedge C)}\ \text{Conj}$$

$$\frac{A \rhd C \qquad B \rhd C}{(A \vee B) \rhd C}\ \text{Disj} \qquad\qquad \frac{A \rhd B \qquad p \in \mathsf{par}}{(p \to A) \rhd (p \to B)}\ \text{Mont}$$

Future Works

1. Relative unification and admissibility for transitive modal logic.
2. Axiomatization or decidability of $\mathrel{|\!\sim}_{\Gamma}$ for Γ being the set of all extendible formulas.
3. Axiomatization or decidability of $\mathrel{|\!\sim}_{\Gamma}$ for Γ being the set of all prime formulas.

Thanks For Your Attention