Completeness proof for IL

Provability logics for relative interpretability by Veltman and De Jongh

March 2021

Preliminaries

The IL logic:

Axioms

- $C_1 : A \to (B \to A)$
- $C_2 : (A \to (B \to C)) \to ((A \to B) \to (A \to C))$
- $C_3 : (\neg A \to \neg B) \to (B \to A)$
- $K : \Box(A \to B) \to (\Box A \to \Box B)$
- $L : \Box(\Box A \to A) \to \Box A$
- $J_1 : \Box(A \to B) \to (A \rhd B)$
- $J_2 : ((A \rhd B) \ \wedge \ (B \rhd C)) \to A \rhd C$
- $J_3 : ((A \rhd C) \wedge (B \rhd C)) \to (A \vee B \rhd C)$
- $J_4 : (A \rhd B) \to (\Diamond A \to \Diamond B)$
- $J_5 : \Diamond A \rhd A$

The IL logic

Rules

- Necessitation: If $\vdash_{IL} A$, then $\vdash_{IL} \Box A$
- Modus Ponnens: If $\Pi \vdash_{IL} A$ and $\Pi \vdash_{IL} A \rightarrow B$, then $\Pi \vdash_{IL} B$.

- Weakening: If $\Pi \vdash_{IL} A$, then $B, \Pi \vdash_{IL} A$
- Structurality: For any substitution σ and $\Pi \vdash_{IL} A$, then $\sigma(\Pi) \vdash_{IL} \sigma(A)$
- Conjunction: $\Pi \vdash_{IL} A \wedge B$ iff $\Pi \vdash_{IL} A$ and $\Pi \vdash_{IL} B$

Example of Hilbert-style proof in IL

$A \rhd A \wedge \Box \neg A$

- $\Box(\Box \neg A \to \neg A) \to \Box \neg A$ — Applying Löbs-axiom to $\neg A$,
- $\Diamond A \to \Diamond \neg(\Box \neg A \to \neg A)$ — By contraposition,
- $\Diamond A \to \Diamond(\Box \neg A \wedge A)$ — By definition,
- $(A \wedge \Diamond A) \rhd \Diamond(\Box \neg A \wedge A)$ — By the $\wedge$-tautology $A \to C \Rightarrow A \wedge B \to C$, necessitation and J1,
- $\Diamond(\Box \neg A \wedge A) \rhd (\Box \neg A \wedge A)$ — Applying J5 to $(\Box \neg A \wedge A)$,
- $A \rhd (A \wedge \Box \neg A) \vee (A \wedge \Diamond A)$ — Since $B \leftrightarrow (B \wedge \neg C) \vee (B \wedge C)$, applying necessitation and J1,
- $((A \wedge \Box \neg A) \vee (A \wedge \Diamond A)) \rhd (A \wedge \Box \neg A) \vee (A \wedge \Box \neg A)$ — Cases,
- $A \rhd (A \wedge \Box \neg A)$ — What we wanted to prove!

Semantics of IL

For an GL-Frame $F = \langle W, R \rangle$ we define $W[u] = \{v \in W |\ uRv\}$

We say that F is an $IL-$frame with an additional relation S_u, for each $u \in W$ with the following properties:

* S_u is reflexive

* S_u is transitive

* for $v,\ w \in W[u]$ if vRw then vS_uw.

Interpretation

An $IL-$model is given by an $IL-$frame $\langle W, R, \{S_u\}_{u \in W} \rangle$ such that

$u \Vdash \Box A$ iff $\forall v \, (uRv \Rightarrow v \Vdash A)$

$u \Vdash A \rhd B$ iff $\forall v \, (uRv \ \wedge \ v \Vdash A \ \Rightarrow \exists w \, (vS_u w \ \wedge \ w \Vdash B))$

Example of proof in IL

$A \rhd A \wedge \Box \neg A$

- $\Box(\Box\neg A \to \neg A) \to \Box\neg A$
- $\Diamond A \to \Diamond\neg(\Box\neg A \to \neg A)$
- $\Diamond A \to \Diamond(\Box\neg A \wedge A)$
- $(A \wedge \Diamond A) \rhd \Diamond(\Box\neg A \wedge A)$
- $\Diamond(\Box\neg A \wedge A) \rhd (\Box\neg A \wedge A)$
- $A \rhd (A \wedge \Box\neg A) \vee (A \wedge \Diamond A)$
- $((A \wedge \Box\neg A) \vee (A \wedge \Diamond A)) \rhd (A \wedge \Box\neg A) \vee (A \wedge \Box\neg A)$
- $A \rhd (A \wedge \Box\neg A)$

Applying Löbs-axiom to $\neg A$,

By contraposition,

By definition,

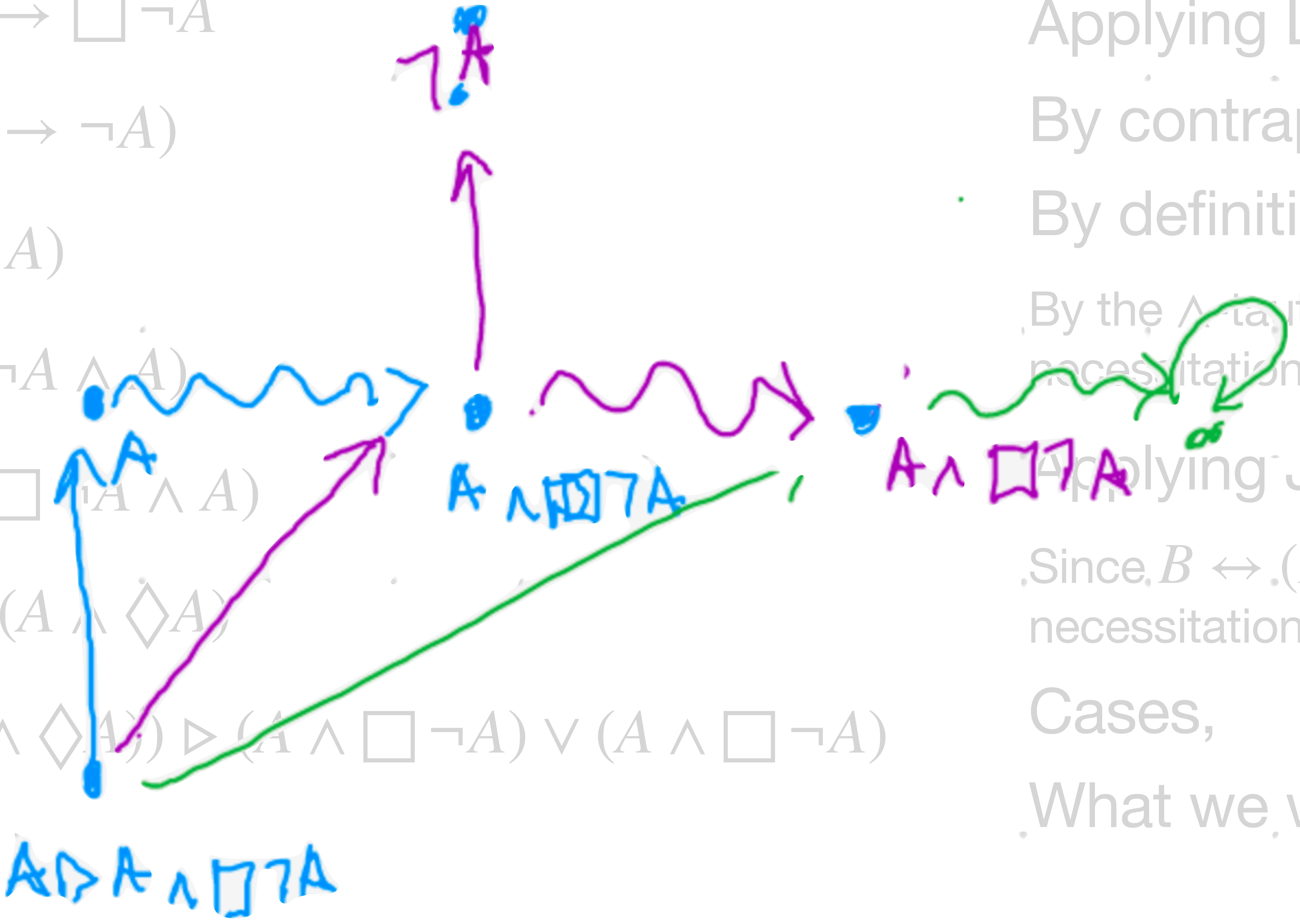
By the $\wedge$-tautology $A \to C \Rightarrow A \wedge B \to C$, necessitation and J1,

Applying J5 to $(\Box\neg A \wedge A)$,

Since $B \leftrightarrow (B \wedge \neg C) \vee (B \wedge C)$, applying necessitation and J1,

Cases,

What we wanted to prove!



Theorem (soundness):

If F is an IL-Frame, for each formula A:

if $\vdash_{IL} A$, then $F \vDash A$.

Completeness theorem for IL
(Preliminaries)

Definition

Adequate sets

A set of formulae Φ is *adequate if:*

(i). Φ is closed under the taking of sub formulae,

(ii). If $B \in \Phi$, and B is no negation of another formula, then $\neg B \in \Phi$,

(iii). $\bot \rhd \bot \in \Phi$,

(iv). If $B \rhd C \in \Phi$, then also $\Diamond B, \Diamond C \in \Phi$,

(v). If B and C are the antecedent or consequent of a $\rhd$-formula in Φ, then $B \rhd C \in \Phi$.

Definition

$\prec$-relation

For Γ and Δ two maximal IL-consistent subsets of formulae of some finite adequate Φ, we say that Δ is a *successor* of Γ, $\Gamma \prec \Delta$ if and only if:

- for each $\Box A \in \Gamma$, then $\Box A, A \in \Delta$
- there is some $\Box A \notin \Gamma$, but $\Box A \in \Delta$

Definition

Let Γ be a maximal IL-consistent subset of some finite adequate Φ, and let W_Γ be the smallest set such that:

(i). $\Gamma \in W_\Gamma$

(ii). If $\Delta \in W_\Gamma$ and Δ' be an IL-consistent subset of Φ such that $\Delta \prec \Delta'$, then $\Delta' \in W_\Gamma$

Lemma

- $\prec$ is transitive and irreflexive on W_Γ
- For each $\Gamma \in W_\Gamma$,

$$\Box A \in \Gamma \text{ if and only if } A \in \Delta \text{, for every } \Delta \text{ such that } \Gamma \prec \Delta$$

Definition

C-critical successors

Let Γ and Δ be maximal IL-consistent subsets of some given adequate Φ. We say that Δ is a C-*critical successor* of Γ if and only if

(i). $\Gamma \prec \Delta$

(ii). $\neg A, \Box \neg A \in \Delta$ for each formula A such that $A \rhd C \in \Gamma$

Note: every successor of Γ is $\bot$-critical successor of Γ.

Lemma

Let Γ be a maximal IL-consistent in Φ

- If $\neg(B \rhd C) \in \Gamma$, there exists a C-critical successor Δ of Γ, maximal IL -consistent in Φ, such that $B \in \Delta$.

- If $B \rhd C \in \Gamma$ and $B \in \Delta$ for some Δ is an D-critical successor of Γ, then there is some Δ', D-critical successor of Γ, such that $C \in \Delta'$.

Completeness theorem for IL

Completeness and decidability of IL

If $\nvdash A$, then there is a finite IL-model K such that $K \nvDash A$

Proof:

Let Φ be some finite adequate set that contains $\neg A$, and Γ_0 be a maximally consistent subset of Φ containing $\neg A$.

We define W_{Γ_0} as the smallest set of pairs such that:

i. $(\Gamma_0, < >) \in W_{\Gamma_0}$, where $< >$ represents the empty sequence.

ii. If $(\Gamma, \tau) \in W_{\Gamma_0}$, then for any Δ such that $\Gamma \prec \Delta$, we have that $(\Delta, \tau) \in W_{\Gamma_0}$.

iii. If $(\Gamma, \tau) \in W_{\Gamma_0}$, then $(\Delta, \tau * < C >) \in W_{\Gamma_0}$ for every C-critical successor.

Completeness and decidability of IL

If $\nvdash A$, then there is a finite IL-model K such that $K \nvDash A$

Proof:

Let Φ be some finite adequate set that contains $\neg A$, and Γ_0 be a maximally consistent subset of Φ containing $\neg A$.

We define W_{Γ_0} as the smallest set of pairs such that:

i. $(\Gamma_0, < >) \in W_{\Gamma_0}$, where $< >$ represents the empty sequence.

ii. If $(\Gamma, \tau) \in W_{\Gamma_0}$, then for any Δ such that $\Gamma \prec \Delta$, we have that $(\Delta, \tau) \in W_{\Gamma_0}$.

iii. If $(\Gamma, \tau) \in W_{\Gamma_0}$, then $(\Delta, \tau * < C >) \in W_{\Gamma_0}$ for every C-critical successor.

***Notation**: for $u = (\Delta, \tau) \in W_{\Gamma_0}$, we denote $(u)_0 = \Delta$ and $(u)_1 = \tau$*

Completeness and decidability of IL

If $\nvdash A$, then there is a finite IL-model K such that $K \nvDash A$

Proof:

What do we know about W_{Γ_0}:

- *It is finite.*
- *If $u \in W_{\Gamma_0}$ and the formula E occurs in the sequence $(u)_1$, then* $\neg E, \Box\neg E \in (u)_0$

Completeness and decidability of IL

If $\nvdash A$, then there is a finite IL-model K such that $K \nvDash A$

Proof:

Definition:

Let $v, w \in W_{\Gamma_0}$, then

vRw *if and only if* $(v)_0 \prec (w)_0$ *, and*

$(v)_1 = (w)_1 * \sigma$*, for some sequence* σ*.*

- ***Claim:*** *R is transitive and Noetherian.*

Completeness and decidability of IL

If $\nvdash A$, then there is a finite IL-model K such that $K \nvDash A$

Proof:

Definition:

Let $u, v, w \in W_{\Gamma_0}$, *then*

vS_uw *if and only if* $(u)_1 = (v)_1 \subseteq (w)_1$, *or*

for some C, σ *and* τ, $(v)_1 = (u)_1 * < C > * \sigma$,

and $(w)_1 = (u)_1 * < C > * \tau$,

- ***Claim:*** S_u *is well defined on* $W_{\Gamma_0}[u]$, *transitive and reflexive relation.*

Completeness and decidability of IL

If $\nvdash A$, then there is a finite IL-model K such that $K \nvDash A$

Proof:

***Definition**: for every proposition variable p, and for $u \in W_{\Gamma_0}$*

$$u \Vdash p \quad \textit{if and only if} \quad p \in (u)_0$$

- ***Claim**: for every formula E,*

$$u \Vdash E \quad \textit{if and only if} \quad E \in (u)_0$$

Completeness and decidability of IL

If $\nvdash A$, then there is a finite IL-model K such that $K \nvDash A$

Proof:

$B \rhd C \in (u)_0$ *iff* $\forall v(uRv \wedge B \in (v)_0 \Rightarrow \exists w(vS_u w \wedge C \in (w)_0))$

$(\Rightarrow)$ *Suppose* $B \rhd C \in (u)_0$. *Consider any* $v \in W_{\Gamma_0}$ *such that* uRv *and* $B \in (v)_0$.

Lemma

Let Γ be a maximal IL-consistent in Φ

- If $\neg(B \rhd C) \in \Gamma$, there exists a C-critical successor Δ of Γ, maximal IL -consistent in Φ, such that $B \in \Delta$.

- If $B \rhd C \in \Gamma$ and $B \in \Delta$ for some Δ is an D-critical successor of Γ, then there is some Δ', D-critical successor of Γ, such that $C \in \Delta'$.

Lemma

Let Γ be a maximal IL-consistent in Φ

- If $\neg(B \rhd C) \in \Gamma$, there exists a C-critical successor Δ of Γ, maximal IL -consistent in Φ, such that $B \in \Delta$.

- If $B \rhd C \in \Gamma$ and $B \in \Delta$ for some Δ is an D-critical successor of Γ, then there is some Δ', D-critical successor of Γ, such that $C \in \Delta'$.

Completeness and decidability of IL

If $\nvdash A$, then there is a finite IL-model K such that $K \nvDash A$

Proof:

$B \rhd C \in (u)_0$ *iff* $\forall v(uRv \wedge B \in (v)_0 \Rightarrow \exists w(vS_u w \wedge C \in (w)_0))$

$(\Rightarrow)$ *Suppose* $B \rhd C \in (u)_0$. *Consider any* $v \in W_{\Gamma_0}$ *such that* uRv *and* $B \in (v)_0$.

Case 1: $(u)_1 * < E > * \tau = (v)_1$, $(u)_0$ *is an* E*-critical successor of* $(w)_0$ *such that* $B \in (u)_0$. *Then there is a* Δ *an* E*-critical successor of* $(v)_0$ *such that* $C \in \Delta$.

Take $w = (\Delta, (u)_0 * < E >)$.

Case 2: $(u)_1 = (v)_1$, *then* $(u)_0 \prec (v)_0$.

Completeness and decidability of IL

If $\nvdash A$, then there is a finite IL-model K such that $K \nvDash A$

Proof:

$B \rhd C \in (u)_0$ *iff* $\forall v(uRv \wedge B \in (v)_0 \Rightarrow \exists w(vS_u w \wedge C \in (w)_0))$

($\Rightarrow$) Suppose $B \rhd C \in (u)_0$. *Consider any* $v \in W_{\Gamma_0}$ *such that* uRv *and* $B \in (v)_0$.

Case 1: $(u)_1 * < E > * \tau = (v)_1$,

Case 2: $(u)_1 = (v)_1$, *then* $(u)_0 \prec (v)_0$. *Then for* $B \rhd C \in (u)_0$, $B \in (u)_0$ *implies that there is a* $\perp$*-critical successor* Δ *of* $(u)_0$ *such that* $C \in \Delta$.

Take $v = (\Delta, (u)_1)$.

Completeness and decidability of IL

If $\nvdash A$, then there is a finite IL-model K such that $K \nvDash A$

Proof:

$B \rhd C \in (u)_0$ *iff* $\forall v(uRv \wedge B \in (v)_0 \Rightarrow \exists w(vS_u w \wedge C \in (w)_0))$

$(\Rightarrow)$ *Suppose* $B \rhd C \in (u)_0$.

$(\Leftarrow)$ *Suppose* $B \rhd C \notin (u)_0$, *then* $\neg(B \rhd C) \in (u)_0$

Lemma

Let Γ be a maximal IL-consistent in Φ

- If $\neg(B \rhd C) \in \Gamma$, there exists a C-critical successor Δ of Γ, maximal IL-consistent in Φ, such that $B \in \Delta$.
- If $B \rhd C \in \Gamma$ and $B \in \Delta$ for some Δ is an D-critical successor of Γ, then there is some Δ', D-critical successor of Γ, such that $C \in \Delta'$.

Lemma

Let Γ be a maximal IL-consistent in Φ

- If $\neg(B \rhd C) \in \Gamma$, there exists a C-critical successor Δ of Γ, maximal IL-consistent in Φ, such that $B \in \Delta$.
- If $B \rhd C \in \Gamma$ and $B \in \Delta$ for some Δ is an D-critical successor of Γ, then there is some Δ', D-critical successor of Γ, such that $C \in \Delta'$.

Completeness and decidability of IL

If $\nvdash A$, then there is a finite IL-model K such that $K \nVdash A$

Proof:

$B \rhd C \in (u)_0$ *iff* $\forall v(uRv \wedge B \in (v)_0 \Rightarrow \exists w(vS_u w \wedge C \in (w)_0))$

($\Rightarrow$) *Suppose* $B \rhd C \in (u)_0$.

($\Leftarrow$) *Suppose* $B \rhd C \notin (u)_0$, *then* $\neg(B \rhd C) \in (u)_0$. *Let* Δ *be a* C*-critical successor of* $(u)_0$ *such that* $B \in \Delta$. *Take* $v = (\Delta, (u)_1 * < C >)$. *For* $w \in W_{\Gamma_0}$ *such that* $vS_u w$, *then* C*-occurs in* $(w)_1$ *which implies that* $\neg C \in (v)_0$.

More axioms…

Other Axioms

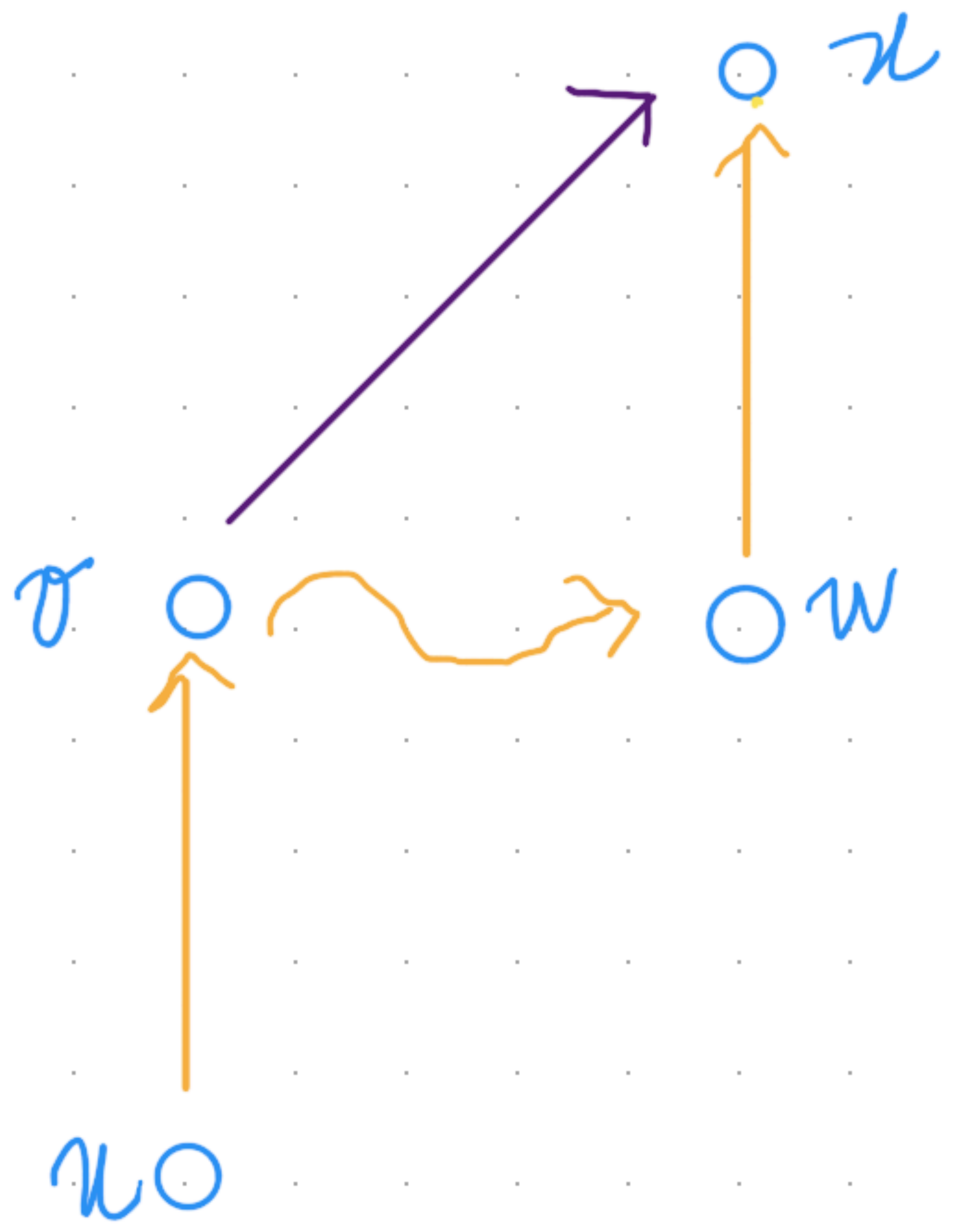
- $M : A \rhd B \to (A \wedge \Box C \rhd B \wedge \Box C)$
- $P : A \rhd B \to \Box (A \rhd B)$
- $W : A \rhd B \to (A \rhd B \wedge \Box \neg A)$

Notation: We write ILS standing for the logic $IL + S$ where S is either axiom M, P, W.

Definition

Let K_S be the family of frames $F = \langle W, R, S_w \rangle$, for which $S \in \{M,\ P,\ W\}$.

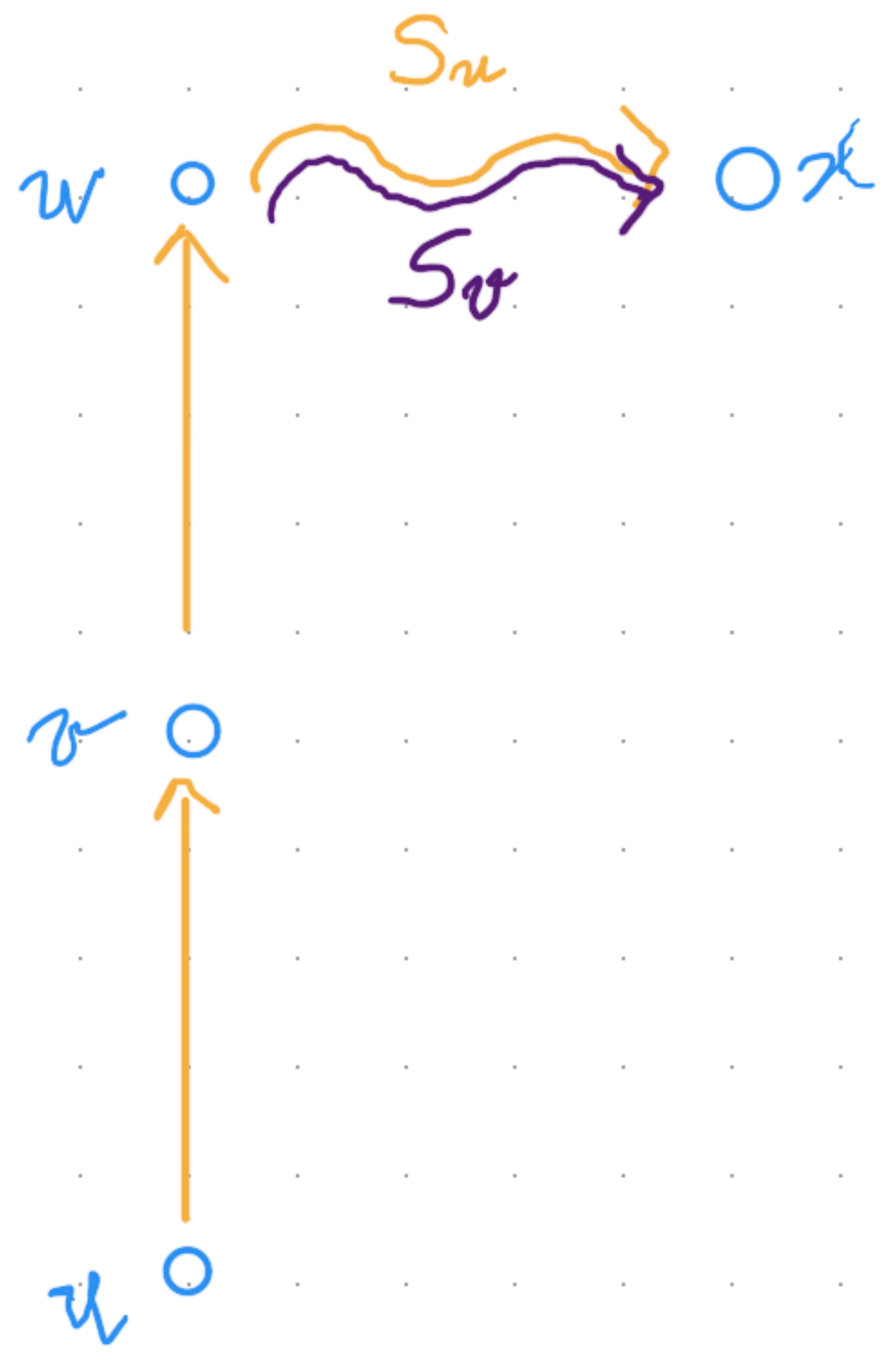
- K_M: for each $u, v, w, x \in W$, if $vS_u wRx$, then vRx.
-
-



Definition

Let K_S be the family of frames $F = \langle W, R, S_w \rangle$, for which $S \in \{M,\ P,\ W\}$.

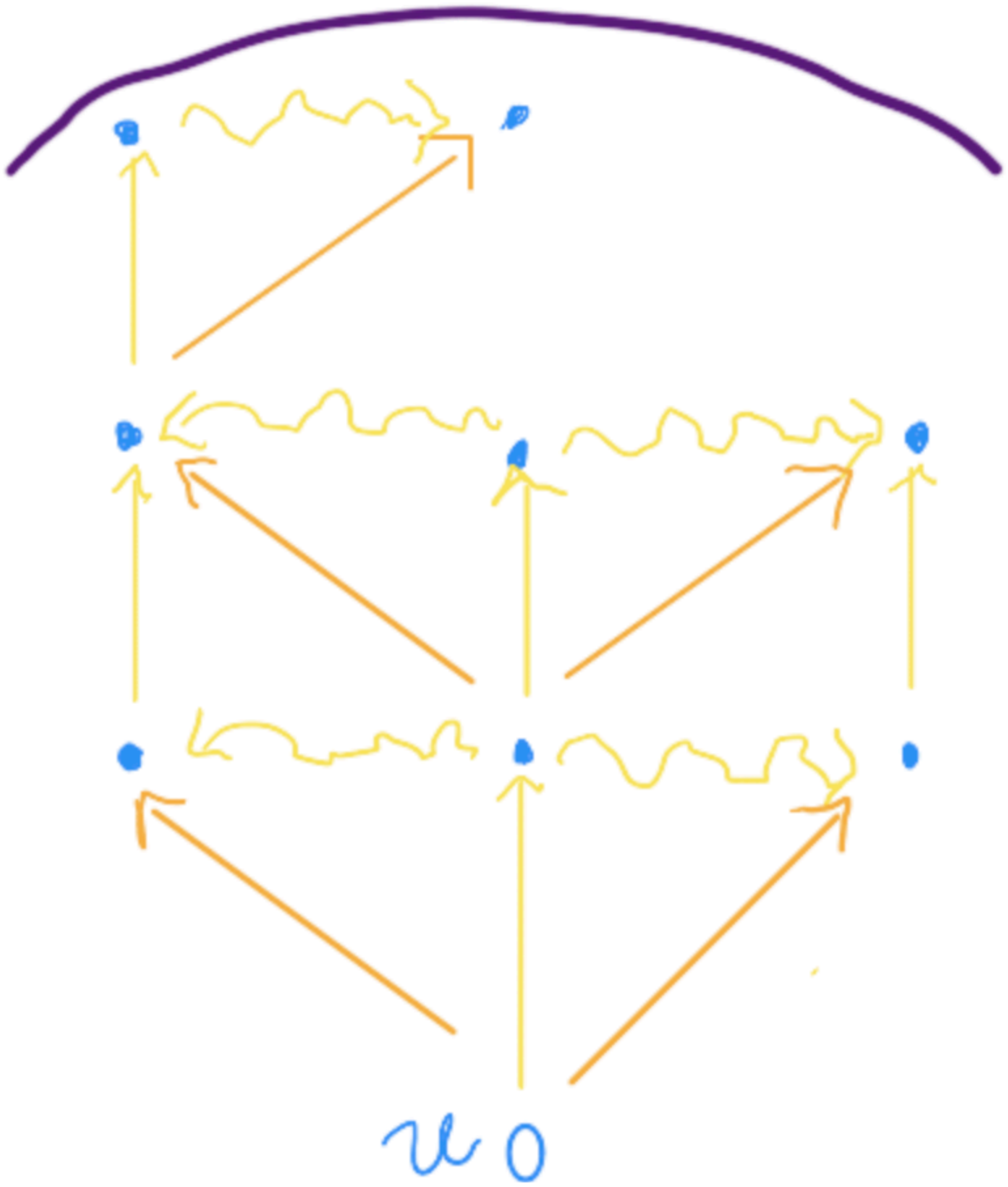
- K_M: for each $u, v, w, x \in W$, if $vS_u wRx$, then vRx.
- K_P : for each $u, v, w, x \in W$, such that uRv and vRw,
  if $wS_u x$ then $wS_v x$.
-



Definition

Let K_S be the family of frames $F = \langle W, R, S_w \rangle$, for which $S \in \{M,\ P,\ W\}$.

- K_M : for each $u, v, w, x \in W$, if $vS_u wRx$, then vRx.
- K_P : for each $u, v, w, x \in W$, such that uRv and vRw, if $wS_u x$ then $wS_v x$.
- K_W : $R \circ S_u$ is conversely well-founded for each $u \in W$.



Theorem (frame conditions):

Let K_S be the family of frames $F = \langle W, R, S_w \rangle$, for which $S \in \{M,\ P,\ W\}$.

- For any frame $F \in K_S$, we have that

$$F \vDash ILS \text{ if and only if } F \in K_S$$

Theorem (soundness):

If F is an IL-Frame, for each formula A:

if $\vdash_{ILS} A$, then $K_S \models A$.

Thanks