Provability logics and applications

Day 1: Provability as modality

1. Give formal proofs to the extend that $\mathbf{K} \vdash \Box A \wedge \Box B \leftrightarrow \Box(A \wedge B)$. (Hints are in the slides.)

2. Let Löb's rule –we write LR– be $\Box A \to A/A$.

   (a) Show that $\mathbf{K} + \mathsf{LR} = \mathbf{K}$

   (b) Show that $\mathbf{K4} + \mathsf{LR} = \mathbf{GL}$

3. Show that $\mathbf{GL} \vdash \Box A \to \Box\Box A$. (Hints are in the slides.)

4. Let λ be Gödel's liar sentence so that $\mathrm{PA} \vdash \neg \mathtt{Prv}_{\mathrm{PA}}(\lambda) \leftrightarrow \lambda$

   (a) Show that $\mathrm{PA} \vdash \lambda \leftrightarrow \mathtt{Con}_{\mathrm{PA}}$.

   (b) Show that if PA is consistent, then $\mathrm{PA} \nvdash \lambda$.

Day 2: Completeness results for GL

1. (a) Exhibit a $\mathbf{GL}$ frame with an increasing chain of length $\omega \cdot 2 + 2$.

   (b) Exhibit a rooted tree where each branch is of finite length but so that there is a point x with $\mathsf{Ord}(x) = \omega$.

   (c) Exhibit a rooted tree where each branch is of finite length but so that there is a point x with $\mathsf{Ord}(x) = \omega \cdot 2$.

   (d) Let FRT be the class of ordinals such that $\alpha \in \mathsf{FRT}$ iff there is some rooted tree T where each branch is of finite length and for some $x \in T$ we have $\mathsf{Ord}(x) = \alpha$. Show that FRT is closed under $\alpha \mapsto \alpha + 1$.

   (e) Show that FRT is closed under addition. That is, if $\alpha \in \mathsf{FRT}$ and $\beta \in \mathsf{FRT}$, then $\alpha + \beta \in \mathsf{FRT}$.

   (f) Show that FRT defines an initial segment of the ordinals, that is, $\alpha \in \mathsf{FRT}$ and $\beta < \alpha$ implies $\beta \in \mathsf{FRT}$.

   (g) Show that FRT is closed under unions.

   (h) Conclude that $\mathsf{FRT} = \mathsf{Ord}$.

   (i) * Determine the size of FRT^{ω} which defined just as FRT but now we require that at each node in the tree only countably many bifurcations/children are allowed.

2. Use the modal completeness theorem to prove

$$\mathbf{GL} \vdash \Box A \quad \Rightarrow \quad \mathbf{GL} \vdash A.$$

3. Prove the generalized fixpoint Lemma:

Lemma 0.1. *If $\psi_1(\vec{x}), \ldots \psi_n(\vec{x})$ are arithmetical formulas where the variables $\vec{x} = \langle x_1, \ldots, x_n \rangle$ appear free, then there are formulas $\psi_1, \ldots \psi_n$ such that, for all $i \leq n$,*

$$\mathsf{PA} \vdash \phi_i \leftrightarrow \psi_i(\ulcorner \dot{\phi}_1 \urcorner, \ldots, \ulcorner \dot{\phi}_n \urcorner).$$

Hint: Use the standard fixpoint lemma and induction on n.

4. With notation as in the proof of Solovay's theorem, show that if f is an arithmetical interpretation with $f(p) = \bigvee_{w \in V(p)}$ and $w \neq 0$ then $\mathfrak{M}, w \models \phi$ if and only if $\mathsf{PA} + \theta(w) \vdash f(\phi)$.

5. In the proof of Solovay's theorem, we defined the formulas $\theta(w)$ using the multiple fixpoint lemma. Write down explicitly the fixpoint equations in terms of the provability predicate $\mathtt{prv}_T$ and the variables $\theta(w), \ulcorner \theta(w) \urcorner$.

Day 3: Polymodal logics

1. Prove that if X is a scattered space then $X \models \Box(\Box p \to p) \to \Box p$.

2. Prove that if $\langle X, \mathcal{T} \rangle$ is a scattered space and $\mathcal{T} \subseteq \mathcal{S}$ then $\langle X, \mathcal{S} \rangle$ is scattered as well.

Day 4: The closed fragment

1. Show that if ϕ, ψ are formulas then $\mathfrak{Ig} \models \Box((\Box\phi \to \Box\psi) \vee (\Box\psi \to \Box\phi))$.

2. Define a valuation V on $\mathfrak{I}g$ such that $\mathfrak{Ig} \models [0]p \wedge \neg[1]p$.

   Conclude that $\mathfrak{Ic}$ is not a valid frame for the full logic GLP_ω.

3. Define a valuation V on $\mathfrak{Ic}$ such that $\mathfrak{Ic} \models \langle 0 \rangle p \wedge \neg[1]\langle 0 \rangle p$.

Day 5: Ordinal analysis

1. Compute the following order-types:

   (a) $o(1)$
   (b) $o(212)$
   (c) $o(100\omega)$
   (d) $o(012101210)$