Provability Logics and Applications
Day 5
Ordinal analysis

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ESSLLI Tutorial, Opole

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- **Proposition** For each ordinal $\alpha < \varepsilon_0$ there is some GLP_ω-worm A such that $T + A$ is Π_1 equivalent to T_α.
- In this lecture: compute order-types of worms

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- The latter contradicts the assumption that $\mathsf{True}_{\Pi_n}(\pi)$

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- By standard techniques in proof-theory this can be lowered to Σ_n induction

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- See how expressible the closed fragment is!

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- This is called the formalized reduction property

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- Let us drop the subscripts $_{\mathrm{EA}}$ in the remainder of this proof

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