Provability Logics and Applications
Day 5
Ordinal analysis

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Worms

Worms: Iterated consistency statements

$$\langle \xi_1 \rangle \langle \xi_2 \rangle \dots \langle \xi_n \rangle \top$$

Worm : the class of all worms

Worm_α : the class of all worms with entries at least α

$$\mathrm{w} <_\xi \mathrm{v} \Leftrightarrow \mathrm{GLP} \vdash \mathrm{w} \to \langle \xi \rangle \mathrm{v}$$

The relation $<_\xi$ is a well-order on Worm_ξ (modulo equivalence).

It is still well-founded on Worm.

Order types

Definition

Given $\mathsf{w} \in \mathsf{Worm}$, we define

$$\mathsf{ot}_{<_0}(\mathsf{w}) = \sup\{\mathsf{ot}_{<_0}(\mathsf{v}) + 1 : \mathsf{v} <_0 \mathsf{w}\}$$

Note: $\sup \varnothing = 0$.

Problem: How to compute $\mathsf{ot}_{<_0}$?

Operations on worms

If w, v are worms, define:

- $w0v$:

$$(\langle a_1\rangle \ldots \langle a_n\rangle \top)0(\langle b_1\rangle \ldots \langle b_m\rangle \top)$$
$$= \langle a_1\rangle \ldots \langle a_n\rangle \langle 0\rangle \langle b_1\rangle \ldots \langle b_m\rangle \top$$

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- $1 \uparrow w$:
$$1 \uparrow \langle a_1\rangle \ldots \langle a_n\rangle \top = \langle 1 + a_1\rangle \ldots \langle 1 + a_n\rangle \top$$

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Today: All worms have natural number entries.

Basic equivalences

Lemma

- *If $n > m$ and ϕ, ψ are formulas then*

$$\mathsf{GLP}_\omega \vdash \langle n \rangle(\phi \wedge \langle m \rangle \psi) \leftrightarrow (\langle n \rangle \phi \wedge \langle m \rangle \psi).$$

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- *If* $\mathsf{w} <_0 \mathsf{v}$ *then* $1 \uparrow \mathsf{w} <_1 1 \uparrow \mathsf{v}$

Decomposing worms

Define $\|w\|$ to be the sum of the length and maximum of w.

Lemma

Every worm $w \neq \top$ is equivalent to one of the form $(1 \uparrow w_1)0w_0$ with $\|w_i\| < \|w\|$.

Key: If $v \neq \top$ then $(1 \uparrow v) \equiv (1 \uparrow v)0\top$ (Exercise)

Note: $w \equiv v$ means that $\mathrm{GLP}_\omega \vdash w \leftrightarrow v$.

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- $o(\top) = 0$
- $o((1 \uparrow w)0v) = o(v) + \omega^{o(w)}$

Correctness of o

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Given worms w, v,

- $o(w) > o(v)$ *implies that* $w >_0 v$
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□

Surjectivity of o

Lemma

The function o is surjective onto ε_0.

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Proof.

Write $\xi = \alpha + \omega^\beta$ and use induction on $\alpha, \beta < \xi$. □

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2. *There is $v <_0 w$ which is inconsistent.*

Corollary

If $o(w) \geq o(v)$ then $v \not>_0 w$.

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There can only be one function $f : Worm \to On$ which is strictly increasing and surjective.

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Proof.

We already know o is strictly increasing and surjective as is ot, so the two are equal. □

Beyond ε_0

The map o readily extends to worms with arbitrary ordinal entries.

The order types we obtain are related to Veblen ordinals.

For example:

- $o(\langle\omega\rangle) = \varepsilon_0$
- $o(\langle\omega^\alpha\rangle) = \varphi_\alpha(0)$
- $o(\langle\Gamma_0\rangle) = \Gamma_0$