Recap of (in)completeness

No non-trivial Kripke models

Beklemishev, Gabelaia: GLP is complete for the class of GLP-spaces

The proof uses non-constructive methods.

Blass: It is consistent with ZFC that the canonical ordinal spaces for GLP_2 are all trivial

Beklemishev: It is also consistent with ZFC that GLP_2 is complete for its canonical ordinal spaces

Bagaria More generally, for all n it is consistent with ZFC that GLP_n has non-trivial canonical ordinal spaces but GLP_{n+1} does not.

The closed fragment

Recall that the closed fragment is written GLP^0 and does not allow propositional variables (only $\perp$).

Beklemishev: GLP^0_ω may be used to perform ordinal analysis of PA, its natural subtheories and some extensions.

Theorem (Ignatiev)

There is a Kripke frame $\mathfrak{Ig}$ such that GLP^0_ω is sound and complete for $\mathfrak{Ig}$.

Ignatiev's model of GLP^0

Given an ordinal $\xi = \alpha + \omega^\beta$, define $\ell\xi = \beta$ ($\ell 0 = 0$).

Ignatiev's model:

$$\mathfrak{Ig} = \langle D, \langle <_n \rangle_{n<\omega} \rangle$$

- $D = \{f : \omega \to \varepsilon_0 : \forall n\, f(n+1) \leq \ell f(n)\}$
- $f <_n g$ if $f(m) = g(m)$ for $m < n$ and $f(n) < g(n)$

Example:

$$\langle \omega^{\omega+1}, \omega, 0, \ldots \rangle <_2 \langle \omega^{\omega+1}, \omega, 1, 0, \ldots \rangle$$

Frame conditions

Ignatiev's model does not satisfy all frame conditions.

$[n]([n]\varphi \to \varphi) \to [n]\varphi$:

$<_n$ is based on an ordinal and hence well-founded

$[n]\varphi \to [n+1]\varphi$:

$$\langle \omega, 1, 0, \ldots \rangle \begin{array}{c} <_1 \\ \not<_0 \end{array} \langle \omega, 0, 0, \ldots \rangle$$

$\langle n \rangle\varphi \to [n+1]\langle n \rangle\varphi$:

$$\frac{\begin{array}{ccc} \langle \omega^\omega, 0, 0, 0, \ldots \rangle & <_1 & \langle \omega^\omega, \omega, 1, 0, \ldots \rangle \\ \langle \omega^\omega, \omega, 0, 0, \ldots \rangle & <_2 & \langle \omega^\omega, \omega, 1, 0, \ldots \rangle \end{array}}{\begin{array}{ccc} \langle \omega^\omega, 0, 0, 0, \ldots \rangle & <_1 & \langle \omega^\omega, \omega, 0, 0, \ldots \rangle \end{array}}$$

The main axis

Definition

A sequence $f : \omega \to \varepsilon_0$ is exact if for all n,

$$f(n+1) = \ell f(n).$$

Main axis: Set of exact sequences E.

Define $\vec{\ell} : \varepsilon_0 \to D$ by

$$\vec{\ell}\xi = \langle \xi, \ell\xi, \ell^2\xi, \ldots, \ell^n\xi, \ldots \rangle$$

Lemma

The function $\vec{\ell}$ is a bijection between ε_0 and E.

Icard topologies

Generalized intervals:

$$(\alpha, \beta)_n = \{f \in D : \alpha < f(n) < \beta\}.$$

Icard defined a structure

$$\mathfrak{Ic} = \langle E, \langle \mathcal{T}_n \rangle_{n<\omega} \rangle.$$

$\mathcal{T}_n$ is generated by intervals of the form

- $E \cap (\alpha, \beta)_m$ for $m < n$
- $E \cap [0, \beta)_m$ for $m \leq n$

Topological conditions

Icard's model does not satisfy all frame conditions either.

$[n]([n]\varphi \to \varphi) \to [n]\varphi$:

$\mathcal{T}_n$ is scattered since $\mathcal{T}_0$ is.

$[n]\varphi \to [n+1]\varphi$: $\mathcal{T}_{n+1}$ is always a refinement of $\mathcal{T}_n$.

$\langle n\rangle\varphi \to [n+1]\langle n\rangle\varphi$: The point

$$\vec{\ell}\omega^{\omega} = \lim_{n\to\omega} \vec{\ell}\omega^{n}$$

should be isolated in $\mathcal{T}_2$.

Simple sets

Definition

A set $S \subseteq D$ is *simple* if there exist finite sets I, J and ordinals $\alpha_{ij}, \beta_{ij}, \sigma_{ij}$ such that

$$S = \bigcup_{i \in I} \bigcap_{j \in J} (\alpha_{ij}, \beta_{ij}]_{\sigma_{ij}}.$$

If all $\sigma_{ij} \leq \lambda$, we say S is λ-*simple*.

Exercise: Simple sets form a Boolean algebra under standard set operations.

Goal: To show that $[\![\phi]\!]$ is always simple.

$$[\![\phi]\!] = \{w : w \Vdash \phi\}$$

Simple sequences

A function $r : \omega \to \varepsilon_0$ is simple if there is N such that $r(n) = 0$ for $n > N$.

Smallest N: last argument of r.

Simple sequences can describe Icard neighborhoods:

- Let $f \in E$, r simple, N the last argument of r
- $r \sqsubseteq f$ if for all $n < N$, $r(n) < f(n)$ and $r(N) \leq f(N)$; $r \sqsubset f$ if also $r(N) < f(N)$
- if $r \sqsubset f$, $B_r(f) = E \cap \bigcap_{n \leq N} (r(n), f(n)]_n$

Note: Every $f \in D$ is simple but not every simple function is in D.

Envelopes

Definition

$$\lceil r \rceil(\xi) = \begin{cases} 0 & \text{for } n > N \\ r(N) & \text{for } n = N \\ r(n) + \omega^{\lceil r \rceil (n+1)} & \text{for } n < N \\ \ell r(n-1) & \text{for } n > N. \end{cases}$$

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- If $r \sqsubseteq f \in D$ then $\lceil r\rceil(n) \leq f(n)$ for $n \leq N$
- If $r \sqsubseteq f \in D$ and $\lceil r\rceil(M) < f(M)$ then for all $m < M$, $\lceil r\rceil(m) < f(m)$.

Diamonds and simple sets

- $S = \bigcap_{n \leq N} (\alpha_n, \beta_n]_n$ a simple set
- r a simple sequence, $r(n) = \alpha_n + 1$ for $n \leq N$, 0 otherwise

Define $S^{(K)} = (\lceil r \rceil(K), \varepsilon_0) \cap \bigcap_{n<K} (\alpha_n, \beta_n]_n$

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1. $E \cap S^{(K)} = d_K S$
2. $S^{(K)} = \{f \in D : \exists g <_K f \textit{ such that } g \in S\}$

Ignatiev vs. Icard

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If $S \neq \varnothing$ is simple then $E \cap S \neq \varnothing$.

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Lemma

If ϕ is closed then $[\![\phi]\!]_{\mathfrak{Ig}}$ is simple and

$$[\![\phi]\!]_{\mathfrak{Ic}} = E \cap [\![\phi]\!]_{\mathfrak{Ig}}\,.$$

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Corollary

$\mathfrak{Ig}$ *and* $\mathfrak{Ic}$ *satisfy the same set of formulas.*

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$[n]\varphi \to [n+1]\varphi$:

$\langle n\rangle\varphi \to [n+1]\langle n\rangle\varphi$: $\square$

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Theorem (Ignatiev, Icard)

GLP^0 *is complete for both* $\mathfrak{Ig}$ *and* $\mathfrak{Ic}$*.*

Concluding remarks

- GL is very nice as a modal logic, but only takes us so far
- GLP is very useful! (Ordinal analysis, ordinal notation systems, unprovable statements...)
- But GLP is tougher to work with
  - No Kripke frames
  - Topological completeness is hard
- The closed fragment gives us a good middle ground
- Here we have Kripke models, simple topological models.
- Our work: Generalize to GLP_Λ