On tame semantics for interpretability logic

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Overview

Motivation

Motivation

In Provability Logic, for a fixed theory T (PL) $\Box A$ reads as

"A" is provable in T.

Interpretability Logic (IL) extends PL adding $A \rhd B$ which means

$$T + A \text{ interprets } T + B$$

We say that S interprets T – $S \rhd T$ – if there exists a mapping

$$j\colon \mathrm{Form}_T \to \mathrm{Form}_S$$

that *preserves structure*, for example, if $\circ$ is a binary logical connective, then $(\varphi \circ \psi)^j = \varphi^j \circ \psi^j$ such that moreover

$$\forall \varphi \Big(\Box_T \varphi \to \Box_S \varphi^j \Big).$$

Example

Natural numbers can be interpreted as sets.

Motivation

We can define the interpretability logic of a theory T.

$$\mathsf{IL}(T) := \{A \mid \forall * \; T \vdash A^*\},$$

where A is a formula in the language $L_{\Box, \rhd}$

$$F := \bot \mid \mathrm{Prop} \mid F \to F \mid \Box F \mid F \rhd F,$$

and $*$ is a translation sending propositional variables to arithmetical sentences.

Motivation

The axioms of the basic interpretability **IL** are

L1 $\Box(A \to B) \to (\Box A \to \Box B)$

L2 $\Box A \to \Box\Box A$

L3 $\Box(\Box A \to A) \to \Box A$

J1 $\boxed{\Box(A \to B) \to A \rhd B}$

J2 $(A \rhd B) \wedge (B \rhd C) \to A \rhd C$

J3 $A \rhd C \wedge B \rhd C \to A \vee B \rhd C$

J4 $A \rhd B \to (\Diamond A \to \Diamond B)$

J5 $\boxed{\Diamond A \rhd A}$

Remark

- J1 tells us that the identity translation yields an interpretation.
- J5 represents Henkin's completeness theorem formalised.

Motivation

There are some interesting principles of interpretability. Namely,

$$\mathsf{M} := A \rhd B \to A \wedge \Box C \rhd B \wedge \Box C \qquad \text{(Montagna)}$$

$$\mathsf{P} := A \rhd B \to \Box(A \rhd B) \qquad \text{(Persistence)}$$

It is known that

$$\mathsf{IL}(\mathsf{PA}) := \mathsf{ILM} \quad \text{(Full induction)}$$

and

$$\mathsf{IL}(I\Sigma_1) := \mathsf{ILP} \quad \text{(Finitely Axiomatized).}$$

Motivation

ILM and ILP motivate the characterisation of IL(All).

$$\mathrm{IL(All)} := \{A \mid \forall T \supseteq I\Delta_0 + \mathrm{Exp}\ \forall * \ T \vdash A^*\},$$

the interpretability logic of al "reasonable" arithmetical theories.

Remark

$\mathrm{IL(All)} \subsetneq \mathrm{ILM} \cap \mathrm{ILP}$

We present some advances on its modal characterization.

Semantics and intersections

Semantics and intersections

In interpretability logic, models are 4-tuples

$$\mathcal{M} := \langle W, R, \{S_x\}_{x\in W}, V\rangle$$

where

- $W \neq \varnothing$
- $R \subseteq W \times W$
- $S_x \subseteq x\upharpoonright \times x\upharpoonright$
- $V : \mathsf{Prop} \to \mathcal{P}(W)$

$x\upharpoonright := \{y \mid xRy\}$.

R transitive and conversely well-founded;

S_x is reflexive, transitive and contains R on $x\upharpoonright$.

$\mathcal{F} = \langle W, R, \{S_x\}_{x\in W}\rangle$ denotes a frame.

Sometimes we denote models as $\mathcal{M} = \langle \mathcal{F}, V\rangle$.

Semantics and intersections

Propositions, implications and *falsum* ($\bot$) are forced as usual.

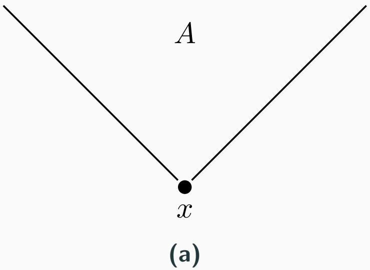
The forcing of formulas $\Box A$ is

$$\mathcal{M}, x \Vdash \Box A :\iff \forall y (xRy \to \mathcal{M}, y \Vdash A).$$

The forcing of formulas $A \rhd B$ is

$$\mathcal{M}, x \Vdash A \rhd B :\iff \forall y (xRy \land \mathcal{M}, y \Vdash A \to \exists z \colon yS_x z \land \mathcal{M}, z \Vdash B).$$

Semantics and intersections



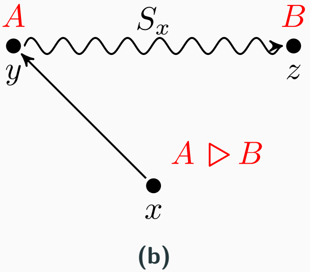


Figure 1: (a) $\Box A$ is forced at x (b) $A \rhd B$ is forced at x

Semantics and intersection

Validity on models and frames is defined as follows.

Validity

- **Validity of a formula on a model:**

$$\mathcal{M} \vDash \varphi \text{ iff } \mathcal{M}, w \Vdash \varphi, \text{ for all } w \in W.$$

- **Validity of a formula on a frame:**

$$\mathcal{F} \vDash \varphi \text{ iff } \forall V \langle \mathcal{F}, V \rangle \vDash \varphi.$$

- **Validity of a scheme:** A model or a frame validates a scheme X ($\mathcal{M} \vDash$ X and $\mathcal{F} \vDash$ X, respectively) iff it validates all X's instances.

Semantics and intersections

The **frame condition** of a scheme X is a first (or higher) order predicate formula $\mathcal{C}$ such that

$$\forall \mathcal{F}(\mathcal{F} \models \mathcal{C} \iff \mathcal{F} \models \mathsf{X}).$$

Example

$$\mathcal{F} \models \Box A \to \Box \Box A \iff \mathcal{F} \models \forall x, y, z \left(xRy \land yRz \to xRz \right)$$

Frame conditions of ILM and ILP.

$$\mathcal{F} \models \mathsf{M} \iff \mathcal{F} \models xRyS_x zRu \to yRu.$$

$$\mathcal{F} \models \mathsf{P} \iff \mathcal{F} \models xRyRzS_x u \to zS_y u.$$

Semantics and intersections

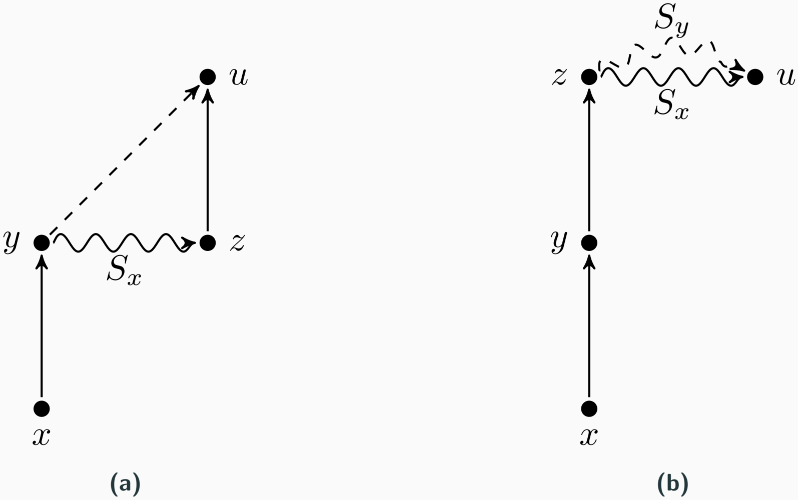


Figure 2: Frame condition of M (a) Frame condition of P (b)

Semantics and intersections

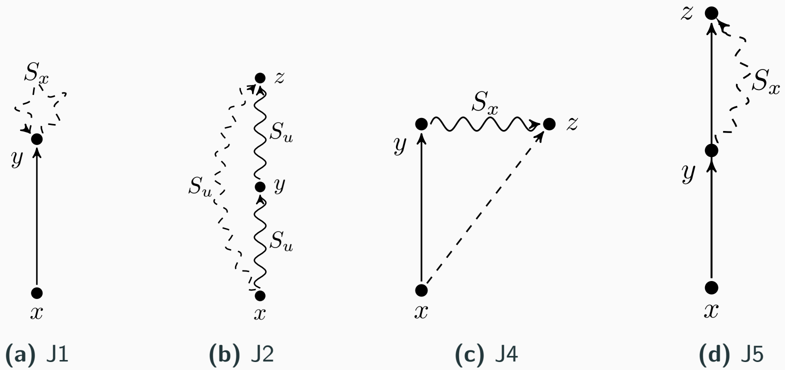


Figure 3: Frame definition reflecting axioms $\Box(A \to B) \to A \rhd B$ (J1), $A \rhd B \wedge B \rhd C \to A \rhd C$ (J2), $A \rhd B \to (\Diamond A \to \Diamond B)$ (J4) and $\Diamond A \rhd A$ (J5)

Semantics and intersections

Sometimes we need to close on the frame properties.

Closure

The closure of a (proto-) frame $\mathcal{F} := \langle W, R, \{S_x\}_{x \in W} \rangle$ under some principle X is the smallest structure $\overline{\mathcal{F}}^X := \langle W, \overline{R}^X, \{\overline{S}_x^X\}_{x \in W} \rangle$ satisfying X such that $R \subseteq \overline{R}^X$ and $S_x \subseteq \overline{S}_x^X$, for every $x \in W$.

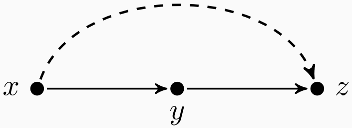


Figure 4: Transitive closure

Semantics and intersection

Frame operator

If $L = \{\phi_i\}_i$ is a set of atomic predicates (like xRy or yS_xz, etc.), we define the **IL-frame induced by** L, $\overline{\mathcal{F}(\bigwedge_i \phi_i)}^{\mathbf{IL}}$, as the universal closure of the smallest proto-frame that satisfies all atomic predicates.

For brevity, we will write $\mathcal{F}(\bigwedge_i \phi_i)$.

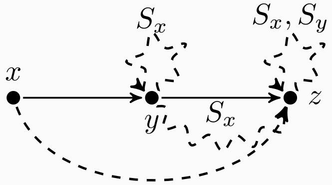


Figure 5: Closure of $\{xRy, yRz\}$ under **IL** frame requirements.

Semantics and intersection

Let $\mathfrak{F}$ be a class of **IL**-frames. We define the interpretability logic corresponding to $\mathfrak{F}$.

$$\mathbf{IL}[\mathfrak{F}] := \{A\colon \text{for all } \mathcal{F} \in \mathfrak{F}, \mathcal{F} \vDash A\}.$$

Let F_{xyz} denote any first or higher order formula where the only free variables are x, y, z.

We now define the following class of conditions.

$$\mathcal{C}_{\mathbf{IL}\mathsf{P} \cap_S \mathbf{IL}\mathsf{M}} :=$$
$$\{F_{xyz} \to xS_y z\colon \mathbf{IL}\mathsf{P} \vDash F_{xyz} \to xS_y z \wedge \mathbf{IL}\mathsf{M} \vDash F_{xyz} \to xS_y z\}.$$

Also, we define the class

$$\mathfrak{All} := \{\mathcal{F} \vDash \mathbf{IL}\mathsf{W}\colon \forall C \in \mathcal{C}_{\mathbf{IL}\mathsf{P} \cap_S \mathbf{IL}\mathsf{M}}, \mathcal{F} \vDash C\}.$$

Semantics and intersection

The principle W is

$$\mathsf{W} := A \rhd B \to A \rhd (B \wedge \Box \neg A)$$

and its frame condition is that there are no $S_x; R$ infinite chains.

Conjecture 1 (Goris, Joosten 2020)

$$\mathbf{IL}(\mathsf{All}) = \mathbf{IL}[\mathfrak{All}].$$

Recall

$$\mathsf{IL}(\mathsf{All}) := \{A \mid \forall T \supseteq I\Delta_0 + \mathsf{Exp}\ \forall *\ T \vdash A^*\}.$$

Semantics and intersection

$\mathsf{M} \cap \mathsf{P}$-closure

Given a proto-frame $\mathcal{F} = \langle W, R, S \rangle$, its $\mathsf{M} \cap \mathsf{P}$-**closure** is $\overline{\mathcal{F}}^{\mathsf{M}\cap\mathsf{P}} := \overline{\mathcal{F}}^{\mathsf{M}} \cap \overline{\mathcal{F}}^{\mathsf{P}} = \langle W, \overline{R}^{\mathsf{M}} \cap \overline{R}^{\mathsf{P}}, \overline{S}^{\mathsf{M}} \cap \overline{S}^{\mathsf{P}} \rangle$.

As an example, consider the principle M_0

$$\mathsf{M}_0 := A \rhd B \to \Diamond A \wedge \Box C \rhd B \wedge \Box C,$$

whose frame condition is

$$\forall x, y, z, u, v \Big(xRyRzS_x uRv \to yRv \Big).$$

Semantics and intersection

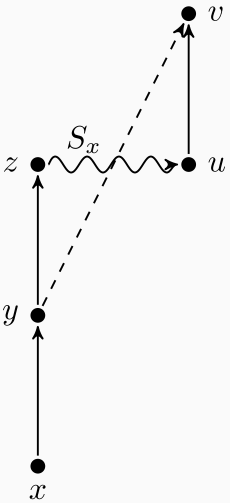


Figure 6: M_0

Semantics and intersection

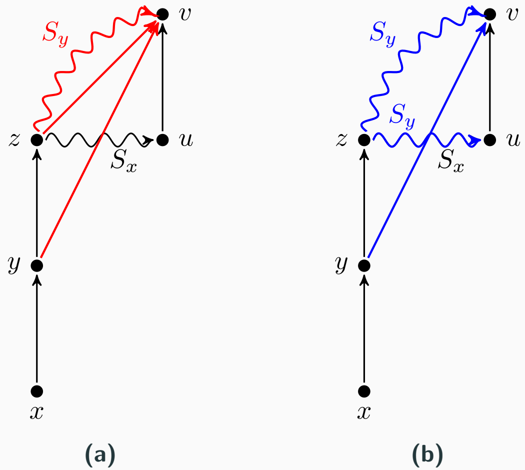


Figure 7: (a) M closure and (b) P closure

Semantics and intersection

$\mathsf{M} \bigcap_{\mathcal{F}} \mathsf{P}$-clause set

We define the $\mathsf{M} \cap_{\mathcal{F}} \mathsf{P}$-clause set as

$$\bigwedge_i \phi_i \to \varphi :\in \mathsf{M} \cap_{\mathcal{F}} \mathsf{P} \text{ iff } \overline{\mathcal{F}(\bigwedge_i \phi_i)}^{\mathsf{M}\cap\mathsf{P}} \models \varphi$$

whenever $\{\phi_i\}_i \cup \{\varphi\}$ is a set of atomic predicates so that $\mathcal{F}(\bigwedge_i \phi_i)$ defines a proto-frame.

Remark

$\bigwedge_i \phi_i \to \varphi$ is a Horn clause.

Non-empty since the M_0 frame condition belongs to it.

It is known the *Broad* series and the *Slim* hierarchy belong to it.

Semantics and intersection

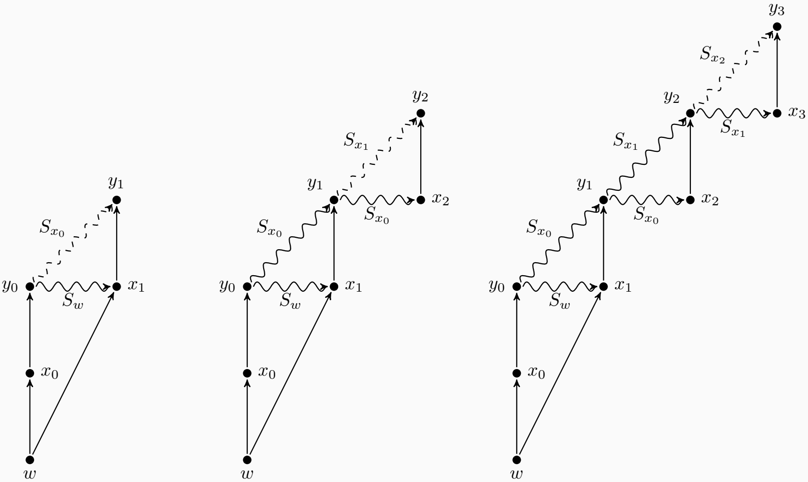


Figure 8: Slim (or Staircase) hierarchy

Semantics and intersection

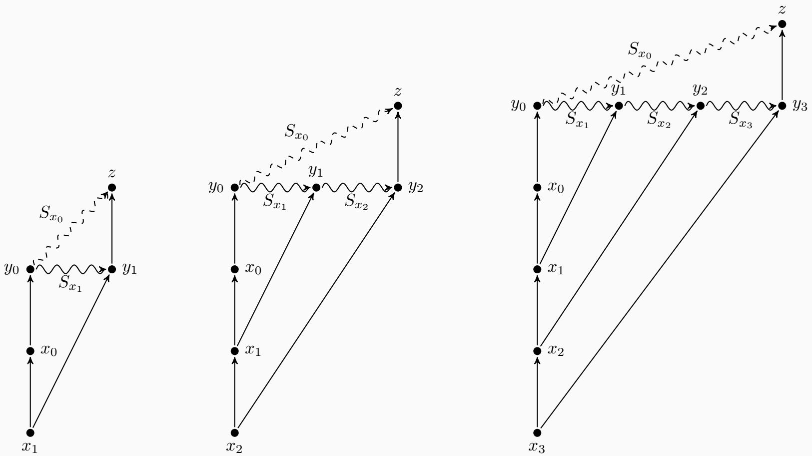


Figure 9: Broad series

Semantics and intersection

$\mathsf{M} \cap_{\mathcal{F}} \mathsf{P}$ defines a fragment of $\mathbf{IL}[\mathfrak{All}]$.

Let us define the lower-case class of **IL**-frames

$$\mathfrak{all} := \{\mathcal{F} \models \mathbf{IL}\mathsf{W} \colon \forall C \in \mathsf{M} \cap_{\mathcal{F}} \mathsf{P}, \mathcal{F} \models C\}.$$

Theorem

$$\mathbf{IL}[\mathfrak{all}] \;\subseteq\; \mathbf{IL}[\mathfrak{All}].$$

Remark

- It is unknown if $\mathbf{IL}[\mathfrak{all}] \;\subset\; \mathbf{IL}[\mathfrak{All}]$.
- $\mathbf{IL}[\mathfrak{all}]$ entails the frame conditions of *Broad* and *Slim*.

Semantics and intersection

It is natural to conjecture that

Conjecture 2

$$\mathbf{IL}[\mathfrak{all}] = \mathsf{IL}(\mathsf{All}).$$

This new conjecture strengthens the old conjecture.

Conjecture 1 (Goris, Joosten 2020)

$$\mathbf{IL}(\mathsf{All}) = \mathbf{IL}[\mathfrak{All}].$$

Semantics and intersection

How can we get a grip on $\mathsf{M} \bigcap_{\mathcal{F}} \mathsf{P}$?

One may try to focus on the clauses that imply an R-pair and conjecture that

Conjecture 3

Consider an **IL**-frame $\mathcal{F} = \langle W, R, S \rangle$. Then, for any $x, y \in W$, we have that $x\overline{R}^{\mathsf{M}} y \wedge x\overline{R}^{\mathsf{P}} y \wedge \neg(xRy) \to x\overline{R}^{\mathsf{M}_0} y$.

Nonetheless, this is disproven by the...

Pencil frame

Pencil frame

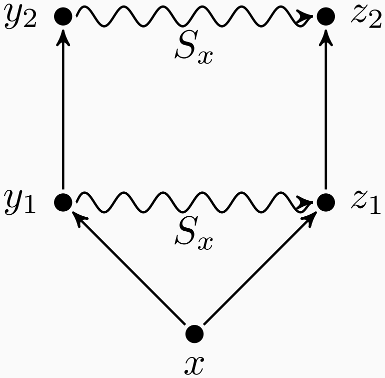


Figure 10: Pencil frame.

Pencil frame

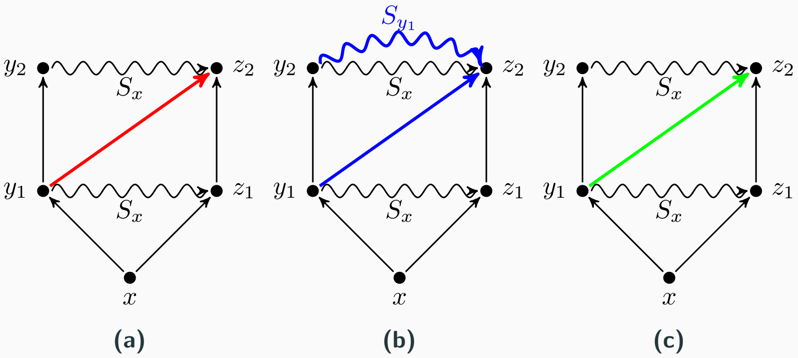


Figure 11: (a) M-closure (b) P-closure (c) Intersection.

Remark

Observe the green arrow is not in the M_0-closure.

Dealing with confluence

On confluence

The unnecessary confluence of Pencil frames hint at their modal undefinability.

Confluence is inherent in interpretability logics (e.g., $xRyS_xz$ implies xRz), but we can unravel **IL**-models into bisimilar *tree-like* models w.r.t. R relations.

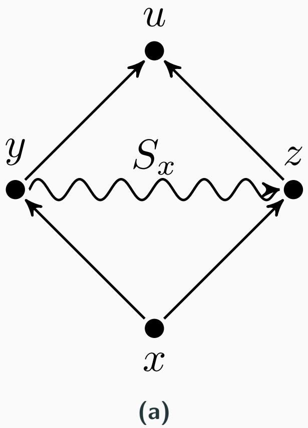
Tree-like IL-model

An **IL**-model $\mathcal{M} = \langle W, R, S, V \rangle$ is **tree-like** if

- (TL1) there exists a unique root regarding R and,
- (TL2) for every world except for the root, there is a immediate unique predecessor regarding R_0

$$xR_0y \text{ iff } xRy \text{ and } \neg\exists z\colon xRzRy$$

On confluence



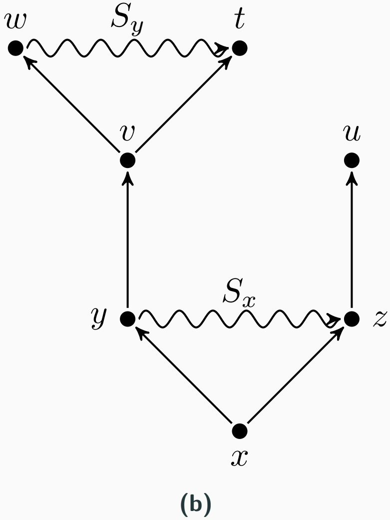


Figure 12: (a) Frame **not** satisfying TL2 (b) Frame satisfying TL2.

On confluence

IL-bisimulation

Two **IL**-models $\mathcal{M} = \langle W, R, S, V \rangle$ and $\mathcal{M}' = \langle W', R', S', V' \rangle$ are **bisimilar**, $\mathcal{M} \underline{\leftrightarrow} \mathcal{M}'$, if there is some $\varnothing \neq Z \subseteq W \times W'$ s.t.:

1. **In**: If wZw', then $w \in V(p)$ iff $w' \in V'(p)$, $\forall p \in \text{Prop}$.
2. **Back:** If wZw' and there is $u \in W$ such that wRu, then there is $u' \in W'$ such that $w'R'u'$ and uZu'. Also, if $u'S'_{w'}v'$, for some $v' \in W'$, then there is $v \in W$ such that $uS_w v$ and vZv'.
3. **Forth:** If wZw' and there is $u' \in W'$ such that $w'R'u'$, then there is $u \in W$ such that wRu and uZu'. Also, if $uS_w v$, for some $v \in W$, then there is $v' \in W'$ such that $u'S'_{w'}v'$ and vZv'.

Use $\mathcal{M}, x \underline{\leftrightarrow} \mathcal{M}', x'$ to indicate that there is a bisimulation connecting x and x'.

On confluence

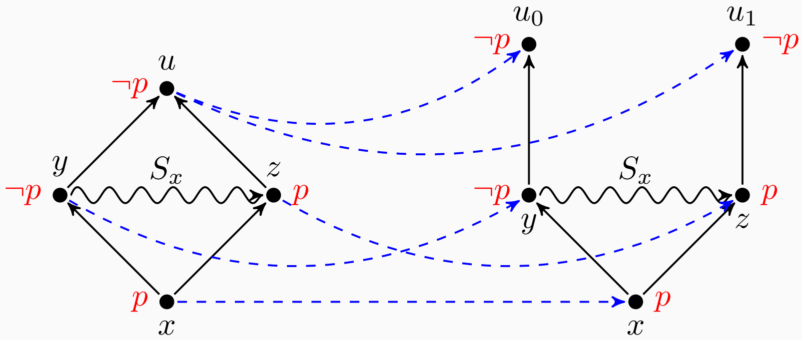


Figure 13: **IL**-bisimulation between two **IL**-models represented in dashed blue arrows.

Key idea

These models *mimic* each other.

On confluence

In fact, bisimilar **IL**-models prove the same modal formulas.

Invariant for bisimulation

A modal formula φ is **invariant under bisimulation** if whenever $x \underline{\leftrightarrow} x'$, then $x \vDash \varphi$ iff $x' \vDash \varphi$.

Of course, we have the following theorem.

Theorem 1

Modal formulas are invariant under bisimulation for **IL**-models.

On confluence

For *restricted* **IL**-models we can find bisimilar tree-like **IL**-models.

Restriction of an IL-model

If $\mathcal{M} = \langle W, R, S, V \rangle$ is an **IL**-model, then its **restriction** to $w \in W$ is a model $\mathcal{M}{\restriction}w = \langle W_{|w}, R_{|w}, S_{|w}, V_{|w} \rangle$ where $W_{|w} = w{\restriction} \cup \{w\}$, $R_{|w} = \{uRv \colon u, v \in W_{|w}\}$, $S_{|w} = \{S_u\}_{u \in W_{|w}}$ and $V_{|w} \colon \text{Prop} \to \mathcal{P}(W_{|w})$.

On confluence

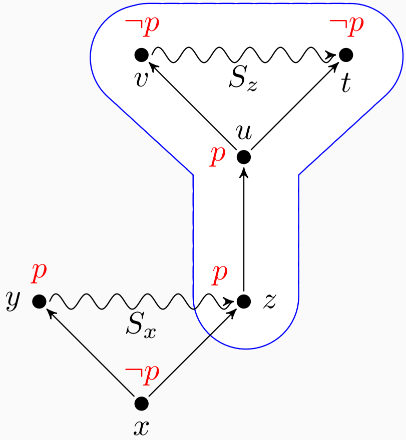


Figure 14: Inclosed in blue: a restricted **IL**-model.

On confluence

Indeed,

Theorem 2

For each **IL**-model $\mathcal{M} = \langle W, R, S, V \rangle$ and world $w_0 \in W$, $\mathcal{M}{\upharpoonright}w_0$ is bisimilar to a tree-like **IL**-model $\mathcal{M}' = \langle W', R', S', V' \rangle$ that is R-wise, in other words, according to the relation R.

Key idea

Use paths.

On confluence

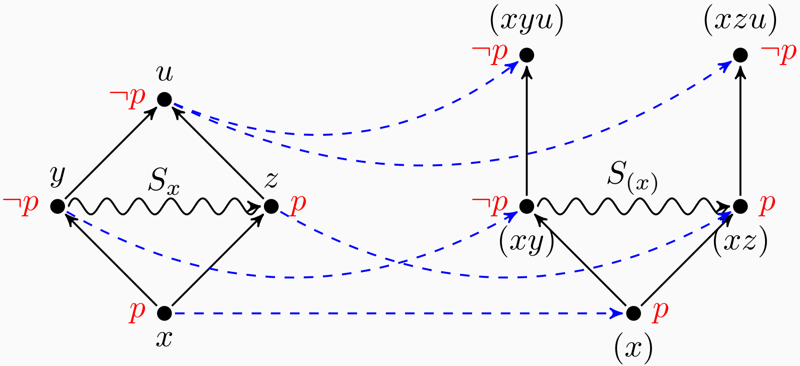


Figure 15: Example of a bisimulation using paths.

A new semantics: Minimal Veltman Semantics

Minimal Veltman Semantics

Let us focus on frames and propose an alternative semantics that avoids certain confluences.

Key idea

To remove many S relations for easier control, with the missing relations compensated by the truth definition of $\rhd$.

We propose the following frames.

Minimal Veltman frames

Consider a non-empty countable set of worlds W, $R \subseteq W \times W$ and, for each $x \in W$, $S_x \subseteq {\upharpoonright}x \times {\upharpoonright}x$ with R transitive and Noetherian; for every $x \in W$, S_x is irreflexive and antitransitive; for every $x, y, z \in W$, $yS_xz \to \neg yRz$. Then, $\mathcal{F} = \langle W, R, \{S_x\}_{x \in W} \rangle$ is a **Minimal Veltman frame** or MV-frame and $\mathcal{C}^{\mathsf{MVF}}$ is the **class of** MV-**frames**.

Minimal Veltman Semantics

Minimal Veltman models

A **Minimal Veltman model** (MV-model) is a tuple $\langle W, R, \{S_x\}_{x \in W}, V \rangle$, where $\langle W, R, \{S_x\}_{x \in W} \rangle$ is an MV-frame, and $V : \mathrm{Prop} \to \mathcal{P}(W)$ is a valuation. The forcing relation $\Vdash_{\mathsf{MV}}$ follows the standard definition for **IL**-models, except for the $\rhd$-modality: [1]

$$\mathcal{M}, x \Vdash_{\mathsf{MV}} A \rhd B \iff \forall y (xRy \Vdash_{\mathsf{MV}} A \to \exists z\, y(R \cup S_x)^* z \Vdash_{\mathsf{MV}} B).$$

$$\mathcal{M}, x \Vdash A \rhd B \iff \forall y (xRy \Vdash A \to \exists z\, yS_x z \Vdash_{\mathsf{MV}} B).$$

[1] Given a binary relation Z, we denote Z^* as the composition of 0 or more copies of Z. We set Z^+ as the composition of 1 or more copies of Z. In fact, $Z^+ = Z; Z^*$.

Minimal Veltman Semantics

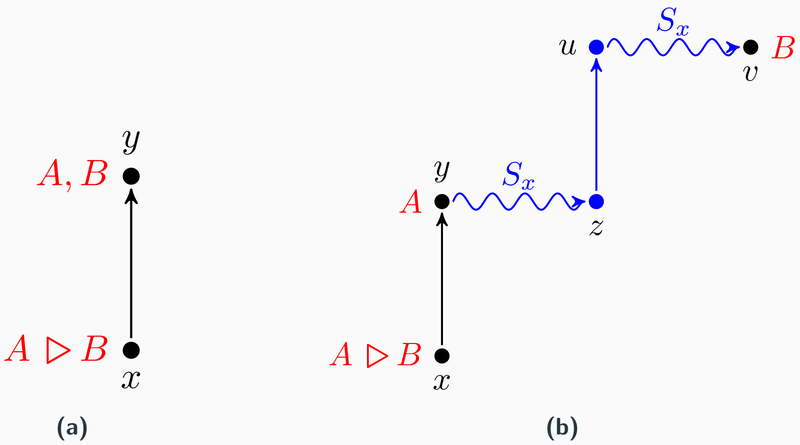


Figure 16: (a) 0 copies of $R \cup S_x$ (b) 3 copies of $R \cup S_x$

Minimal Veltman Semantics

Remark

- We denote the **class of** MV-**models** as $\mathcal{C}^{\mathsf{MVM}}$.
- Validity for formulas and schemes in MV-models and MV-frames is defined as usual, using $\Vdash_{\mathsf{MV}}$ and $\vDash_{\mathsf{MV}}$ for forcing and consequence, respectively.

IL is sound wrt the Minimal Veltman Semantics (MVS).

Theorem 3

$$\mathbf{IL} \vdash \varphi \Rightarrow \forall \mathcal{F} \in \mathcal{C}^{\mathsf{MVF}} \mathcal{F} \vDash_{\mathsf{MV}} \varphi.$$

Completeness of Minimal Veltman Semantics

MVS **completeness**

We can define bisimulations between **IL**-models and MV-models.

Bisimulation between IL-models and MV-**models**

An **IL**-model $\mathcal{M} = \langle W, R, S, V \rangle$ and an MV-model $\mathcal{M}' = \langle W', R', S', V' \rangle$ are **bisimilar**, $\mathcal{M} \underline{\leftrightarrow} \mathcal{M}'$, if there is some $\varnothing \neq Z \subseteq W \times W'$ such that:

1. **In**: If wZw', then $w \in V(p)$ iff $w' \in V'(p)$, $\forall p \in \text{Prop}$.
2. **Back:** If wZw' and there is $u \in W$ such that wRu, then there is $u' \in W'$ with $w'R'u'$ and uZu'. Also, if $u'(R' \cup S'_{w'})^* v'$, for some $v' \in W$, then there is $v \in W$ such that $uS_w v$ and vZv'.
3. **Forth:** If wZw' and there is $u' \in W'$ such that $w'R'u'$, then there is $u \in W$ such that wRu and uZu'. Also, if $uS_w v$, for some $v \in W$, then there is $v' \in W'$ such that $u'(R' \cup S'_{w'})^* v'$ and vZv'.

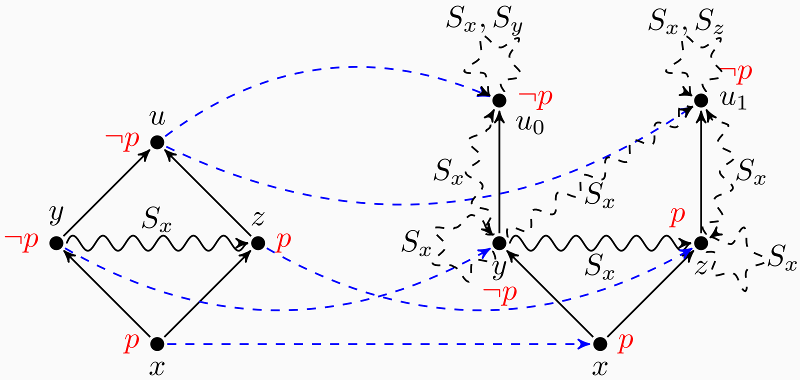


Figure 17: The model on the right is an MV-model. The dashed arrows are the missing arrows.

MVS **completeness**

As always,

Theorem 5

Modal formulas are invariant under bisimulation between **IL**- and MV-models.

and we can prove that

Theorem 6

Each **IL**-model is bisimilar to a MV-model.

Key idea

Remove S-relations incompatible with the MV-frame definition.

MVS **completeness**

Completeness

$\forall \mathcal{F} \in \mathcal{C}^{\mathsf{MVM}} \mathcal{F} \models_{\mathsf{MV}} \varphi \Rightarrow \mathbf{IL} \vdash \varphi$.

Overview of the proof.

- Assume $\mathbf{IL} \nvdash \varphi$.
- By **IL**-completeness, there is an **IL**-model such that $\mathcal{M}, w \nVdash \varphi$.
- $\mathcal{M} \underline{\leftrightarrow} \mathcal{M}'$ for some MV-model $\mathcal{M}'$. (**Thm. 6**)
- $\mathcal{M}$ and $\mathcal{M}'$ prove the same modal formulas. (**Thm. 5**)
- Thus, $\mathcal{M}' \not\models_{\mathsf{MV}} \varphi$.

$\square$

Confluence revisited

Confluence revisited

Avoiding confluence in frames is desirable. We present the (non-trivial) *Unique Path Condition*.

$$(uR^*aS_xbR^*v \land uR^*cS_xdR^*v) \rightarrow \langle a, b\rangle = \langle c, d\rangle. \qquad \text{(UPath)}$$

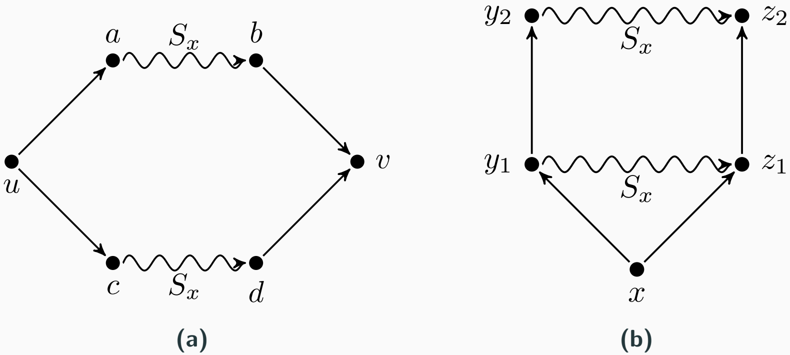


Figure 18: (a,b) Examples of frames that **do not** satisfy UPath

Confluence revisited

That motivates the definition of a new type of models that avoids confluences.

Tree-some (not threesome) models

A rooted MV-model is **tree-some** if it satisfies (UPath).

Confluence revisited

Also, we can define bisimulations between MV-models.

MV-bisimulation

Two MV-models $\mathcal{M} = \langle W, R, S, V \rangle$ and $\mathcal{M}' = \langle W', R', S', V' \rangle$ are MV-**bisimilar** ($\mathcal{M} \underline{\leftrightarrow}_{\mathsf{MV}} \mathcal{M}'$) if there is a $\varnothing \neq Z \subseteq W \times W'$:

1. **In**: If wZw', then $w \in V(p)$ iff $w' \in V'(p)$, $\forall p \in \mathsf{Prop}$.
2. **Back:** If wZw' and there exists $u \in W$ such that wRu, then there exists $u' \in W'$ such that $w'R'u'$ and uZu'. Also, if $u'(R' \cup S'_{w'})^* v'$, for some $v' \in W$, then there exists $v \in W$ such that $u(R' \cup S_w)^* v$ and vZv'.
3. **Forth:** If wZw' and there exists $u' \in W'$ such that $w'R'u'$, then there exists $u \in W$ such that wRu and uZu'. Also, if $u(R' \cup S_w)^* v$, for some $v \in W$, then there exists $v' \in W'$ such that $u'(R' \cup S'_{w'})^* v'$ and vZv'.

Confluence revisited

Of course, modal formulas are invariant under MV-bisimulation.

Theorem 7

Modal formulas are invariant under MV-bisimulation.

Theorem 8

For each MV-model $\mathcal{M} = \langle W, R, S, V \rangle$ and each world $w_0 \in W$ we have that $\mathcal{M} \restriction w_0$ is bisimilar to a tree-some MV-model $\mathcal{M}' = \langle W', R', S', V' \rangle$.

Key idea

Use (more complicated) labeled paths.

This fact allows us to avoid the Pencil model, but Slim and Broad series work with tree-some frames.

Confluence revisited

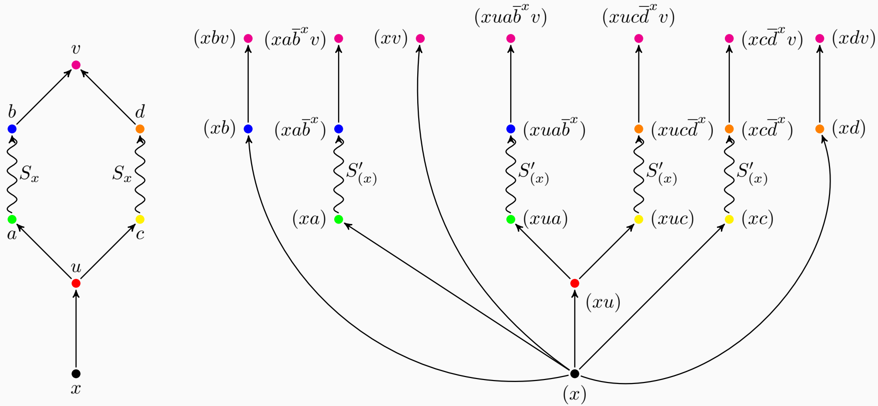


Figure 19: Full unraveling. Bisimilarity indicated by colours.

Summary (Summer-e) and Further work

All in all,

- We strengthen the old conjecture by focusing on Horn clauses;
- We show that all known principles fall in this class;
- We addressed unnecessary confluence via unraveling techniques over tree-like structures;
- We define a new semantics to mitigate confluences and proved its soundness and completeness using bisimulation.

There is still some work to do.

- Explore if every (restricted) **IL**M- or **IL**P-model admits a bisimilar (tree-some) MV-model satisfying the corresponding M- or P-frame condition.

Farewell

Thank you! Obrigado!