

Completeness and Incompleteness for Expanding Gödel-Löb Logics

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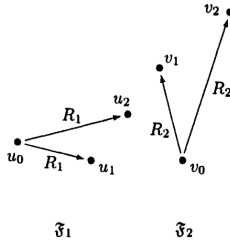
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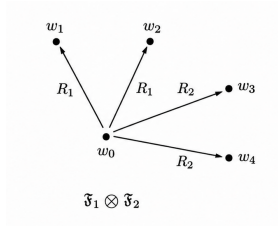
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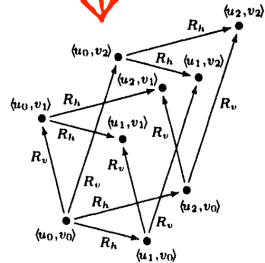
How to combine modal logics?



fusion of
modal logics



product of
modal logics



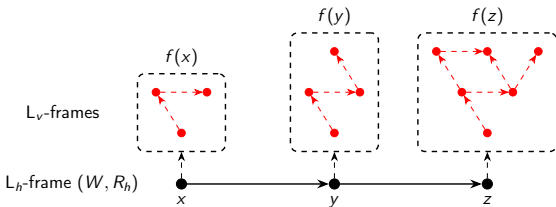
$$\mathfrak{F}_1 \times \mathfrak{F}_2$$

Product logics with expanding domains

A $(L_h \times L_v)^e$ expanding domain frame or e-frame is a pair $(\mathfrak{F}, f)$ st

- $\mathfrak{F} = (W, \prec_h)$ is an L_h -frame
- $f(x) = (W^x, \prec_v^x)$ is an L_v -frame for $x \in W$ st

$$x \prec_h y \Rightarrow f(x) \subseteq_{\text{subframe}} f(y)$$

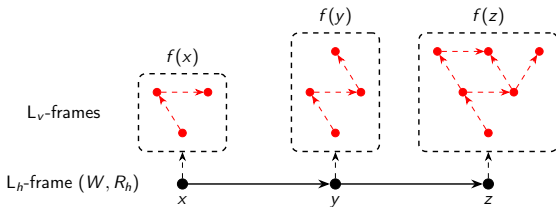


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- $\mathfrak{M}, (x, u) \Vdash \Diamond_h \varphi$ iff $\exists y \in W$ st $x \prec_h y$ and $\mathfrak{M}, (y, u) \Vdash \varphi$,
- $\mathfrak{M}, (x, u) \Vdash \Diamond_v \varphi$ iff $\exists w \in W^x$ st $u \prec_v^x w$ and $\mathfrak{M}, (x, w) \Vdash \varphi$.

Expanding commutator

$[L_h, L_v]^e$ is the least logic over $\mathcal{L}_{hv}$ containing

- all ‘horizontal instances’ of formulas of L_h ,
- all ‘vertical instances’ of formulas of L_v and
- the expanding product axiom schemes

$$\Diamond_v \Diamond_h p \rightarrow \Diamond_h \Diamond_v p \quad (\text{Left commutativity})$$

$$\Diamond_v \Box_h p \rightarrow \Box_h \Diamond_v p \quad (\text{Church–Rosser})$$

Theorem ([Kurucz et al. 03])

$[L_h, L_v]^e$ is complete for

- $L_h \in \{K, K4, S4, S5\}$ and
- L_v axiomatisable by formulas with a universal Horn first-order translation.

Theorem ([Gabelaia et al. 06])

The expanding domain product of many transitive logics is decidable (despite standard products not being so).

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Theorem ([Gabelaia et al. 06])

The expanding domain product of many transitive logics is decidable (despite standard products not being so).

*but ... for which expanding products is the commutator complete?
which Noetherean products are complete?*

Forward-confluent semantics

A frame $L_h \times L_v$ is a bi-relational frame $(W, \prec_h, \prec_v)$ where

- $(W, \prec_h)$ is an L_h -frame
- $(W, \prec_v)$ is an L_v -frame
- the following conditions are satisfied

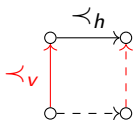


Figure: LC

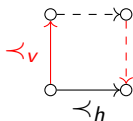


Figure: CR

Lemma

Satisfiability in $(L_h \times L_v)^e \Rightarrow \text{satisfiability in } L_h \times L_v$.

Proposition

$[L_h, L_v]^e$ is sound for $L_h \times L_v$.

Theorem

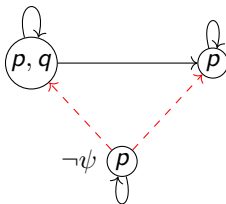
$[\text{Grz}, \text{K}]^e$, $[\text{Grz}, \text{K4}]^e$, and $[\text{Grz}, \text{GL}]^e$ are incomplete.

Proof.

Let $\Box\phi = \phi \wedge \Box\phi$. Consider the formula

$$\psi := p \wedge \Box_h(\Box_v p \vee \Box_v \neg p) \rightarrow \Diamond_h(p \wedge \Box_v(q \rightarrow \Box_h(p \rightarrow q))).$$

ψ is valid over $(\text{Grz} \times \text{K})^e$. Consider the bi-relational model $\mathcal{M}_1$:



$\mathcal{M}_1$ is a forward-confluent model and $\mathcal{M}_1 \not\models \psi$. By soundness, $\not\models \psi$.

Theorem

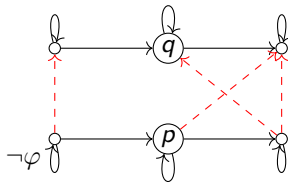
$[\text{Grz.3}, \text{GL}]^e$, $[\text{K4.3}, \text{GL}]^e$, and $[\text{K4.3}, \text{K}]^e$ are incomplete.

Proof.

Let $\Diamond\phi := \phi \vee \Diamond\phi$. Consider the formula

$$\varphi := \Diamond_h p \wedge \Diamond_v \Diamond_h q \rightarrow \Diamond_h (p \wedge \Diamond_v \Diamond_h q) \vee \Diamond_h (\Diamond_h p \wedge \Diamond_v q).$$

φ is valid over $(\text{K4.3} \times \text{K})^e$. Consider the bi-relational model $\mathcal{M}_2$:



$\mathcal{M}_2$ is a forward-confluent model and $\mathcal{M}_2 \not\models \varphi$. By soundness, $\not\models \varphi$.

Theorem

The expanding commutator $[K4, GL]^e$ is sound and complete wrt $(K4 \times GL)^e$ and $K4 \rtimes GL$.

Theorem

The expanding commutator $[GL, GL]^e$ is sound and complete wrt $(GL \times GL)^e$ and $GL \rtimes GL$.

The canonical forward-confluent model

$\mathcal{M}^c := (W^c, \prec_h^c, \prec_v^c, \|\cdot\|^c)$ where

- W^c is the set of all maximally consistent sets in $[L_h, L_v]^e$,
- $\Phi \prec_\alpha^c \Psi \iff \forall \varphi \in \mathcal{L}_{hv} \ (\varphi \in \Psi \Rightarrow \Diamond_\alpha \varphi \in \Phi)$ for $\alpha \in \{h, v\}$,
- $\|p\|^c := \{\Phi \in W^c \mid p \in \Phi\}$.

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Lemma (Truth Lemma)

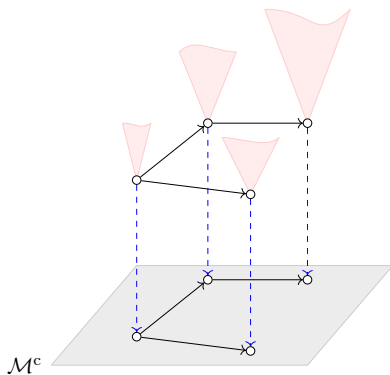
$\varphi \in \Phi \iff \mathcal{M}^c, \Phi \Vdash \varphi$.

Lemma (GL-Witness property)

Let $GL \subseteq L_\alpha$. If $\Diamond_\alpha \varphi \in \Phi$, then $\exists \Psi \in W^c$ st $\Phi \prec_\alpha^c \Psi$ and $\varphi \wedge \neg \Diamond_\alpha \varphi \in \Psi$.

In the following, $\mathcal{M}^c$ is of $[K4, GL]^e$ or $[GL, GL]^e$.

Quasimodel extraction: first intuition



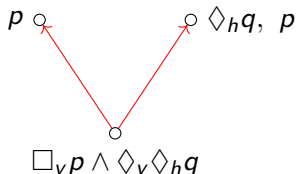
Let $\Sigma \subseteq \mathcal{L}_{hv}$ closed under subformulas and single negations.

A Σ -moment is a structure

$$m := (|m|, \prec_v^m, \ell^m)$$

where

- $(|m|, \prec_v^m)$ is a finite tree-like strict partial order,
- $\ell^m: |m| \rightarrow T_\Sigma$ finite and respects the GL semantics of $\Diamond_v$.



$m \prec_h^\Sigma n$ (*horizontal successor*) := m embeds into n respecting the semantics of transitive $\Diamond_h$.

The moment of Kruskal

Theorem ([Kruskal60])

For any finite set Λ , the class of finite Λ -labelled trees is well-quasi-ordered under embeddability, i.e.,

$(\mathfrak{m}_n)_{n < \omega}$ is an infinite sequence of Σ -moments $\Rightarrow \exists i < j < \omega$ st $\mathfrak{m}_i \sqsubseteq^\Sigma \mathfrak{m}_j$ preserving root and labels.

Corollary

The set of Σ -moments is well-quasi-ordered.

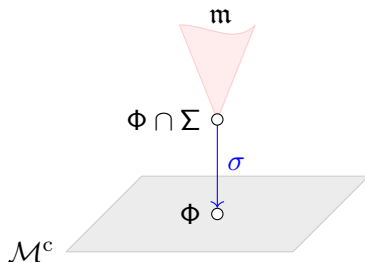
Simulations

We extract quasimodels from the canonical model by relating moments and maximal consistent sets.

A simulation is a map $\sigma: |\mathfrak{m}| \rightarrow W^c$ st

- σ is a homomorphism wrt $\prec_v$,
- $\ell^{\mathfrak{m}}(w) = \sigma(w) \cap \Sigma$ for all $w \in |\mathfrak{m}|$.

$\mathfrak{m}$ *simulates* Φ iff $\exists \sigma: |\mathfrak{m}| \rightarrow W^c$ st $\sigma(r^{\mathfrak{m}}) = \Phi$, denoted $\mathfrak{m} \rightarrow \Phi$.



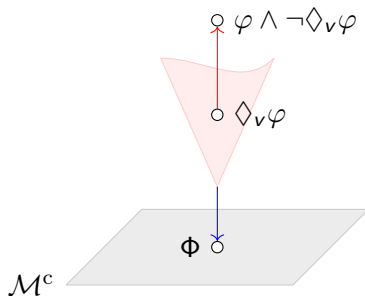
Simulation existence

Lemma

There exists a Σ -moment $\mathfrak{m}$ such that $\mathfrak{m} \rightarrow \Phi$ for any $\Phi \in W^c$.

Proof.

Define $(\mathfrak{m}_i)_{i \in \mathbb{N}}$ by adding vertical witnesses.



$(\mathfrak{m}_i)_{i \in \mathbb{N}}$ stabilizes at $\mathfrak{m} \rightarrow \Phi$.

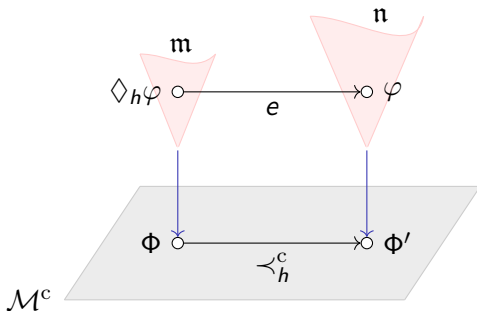
Horizontal realizability

Lemma

Let $m \multimap \Phi$ having a *horizontal defect* $\Diamond_h \varphi \in \ell^m(w)$.

Then, $\exists n$ a moment and $\exists \Phi' \succ_h^c \Phi$ st

- $n \multimap \Phi'$,
- n realizes the horizontal defect, $n \succ_h^\Sigma m$ is horizontal successor providing witness to the defect.

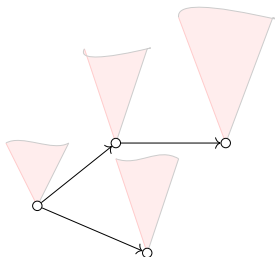


$\mathfrak{Q} := (T^{\mathfrak{Q}}, \prec_h^{\mathfrak{Q}}, \mu^{\mathfrak{Q}})$ where

- $(T^{\mathfrak{Q}}, \prec_h^{\mathfrak{Q}})$ is a transitive irreflexive tree,
- $\mu^{\mathfrak{Q}} : T^{\mathfrak{Q}} \rightarrow \Sigma\text{-Moments}$ **preserving horizontal succession:**

$$x \prec_h^{\mathfrak{Q}} y \Rightarrow \mu^{\mathfrak{Q}}(x) \prec_h^{\Sigma} \mu^{\mathfrak{Q}}(y)$$

- **every horizontal defect is realized.**



Lemma

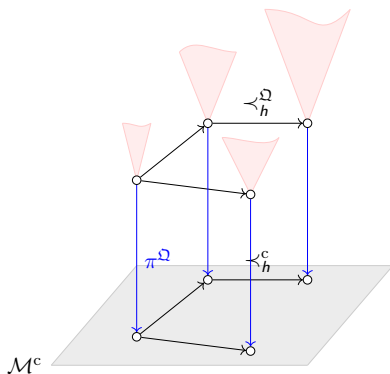
- *Satisfiability in $\mathfrak{Q} \Rightarrow$ satisfiability in a $(K4 \times GL)^e$ -model.*
- *If $\mathfrak{Q}$ is **finite** $\Rightarrow$ satisfiability in a $(GL \times GL)^e$ -model.*

Morphisms

A morphism is essentially a collection of simulations relating a quasimodel to the canonical model.

A Ω -morphism is a map $\pi^\Omega: T^\Omega \rightarrow W^c$ st

- is a homomorphism wrt $\prec_h$ and
- $\mu^\Omega(t) \rightarrow \pi^\Omega(t) \forall t \in T^\Omega$.



Lemma

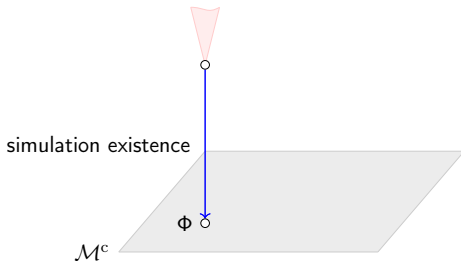
Let $\mathcal{M}^c$ be the canonical model of $[K4, GL]^e$ and $\Phi \in W^c$.
There exists $\mathfrak{Q}$ and $\pi^{\mathfrak{Q}}$ satisfying $\pi^{\mathfrak{Q}}(r^{\mathfrak{Q}}) = \Phi$.

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Proof.

Define $\{(\mathfrak{Q}_i, \pi^i)\}_{i \in \mathbb{N}}$.



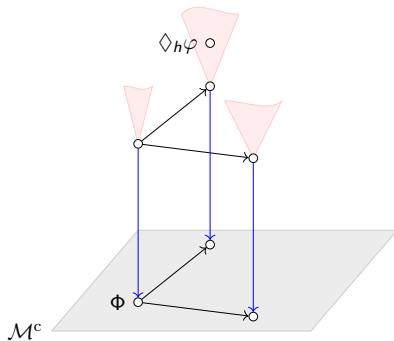
Quasimodel extraction

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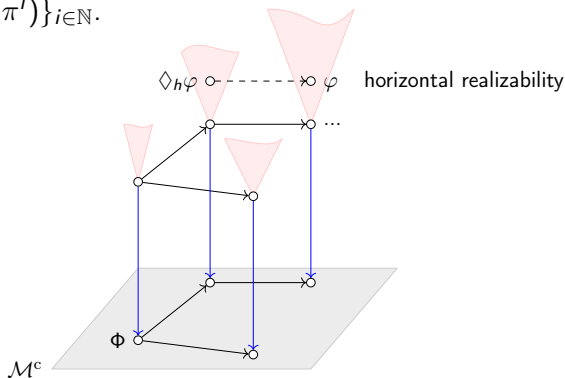
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Proof.

Define $\{(\mathfrak{Q}_i, \pi^i)\}_{i \in \mathbb{N}}$.



The direct limit $(\mathfrak{Q}_\infty, \pi^\infty)$ satisfies the statement.

Completeness of $[K4, GL]^e$

Theorem

The logic $[K4, GL]^e$ is sound and complete wrt $(K4 \times GL)^e$ and $K4 \times GL$.

Proof.

(Completeness): Let φ consistent $\Rightarrow \varphi \in \Phi$ a MCS.

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- 2 Since $\mu^\Omega(r^\Omega) \rightarrow \Phi$, by definition φ is in the label of the root of $\mu^\Omega(r^\Omega)$, i.e. φ is satisfied in Ω .
- 3 Thus, φ is satisfied in a $(K4 \times GL)^e$ -model.



Yankov-Fine formulas for $[GL, GL]^e$

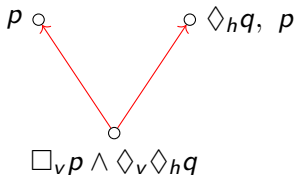
Yankov-Fine formulas internalize the notion of simulation into the modal language.

$$\Box_v p \wedge \overset{\circ}{\Diamond}_v \Diamond_h q$$

$$\text{Sim}(m) = \Box_v p \wedge \Diamond_v \Diamond_h q$$

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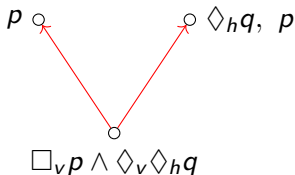
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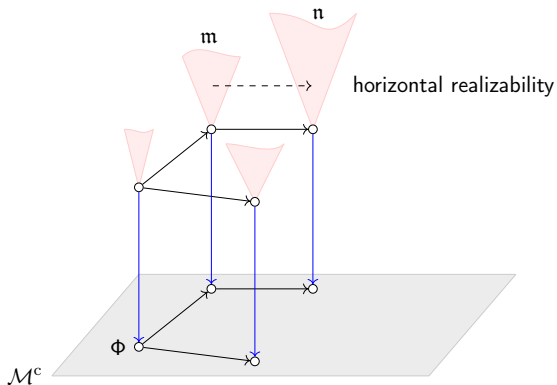
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Theorem

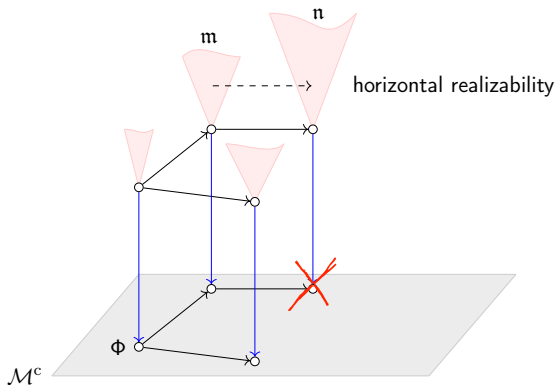
For any Σ -moment $\mathfrak{m}$ and $\Phi \in W^c$,

$$\mathfrak{m} \rightarrow \Phi \iff \text{Sim}(\mathfrak{m}) \in \Phi.$$

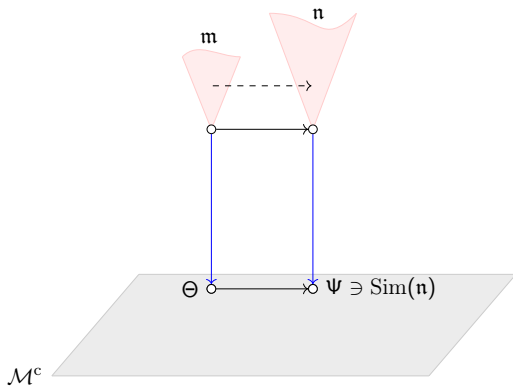
Refining quasimodel extraction with Yankov-Fine



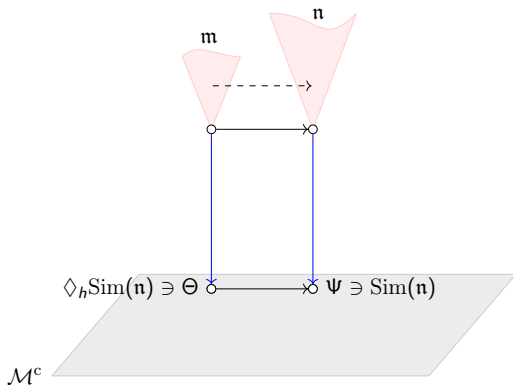
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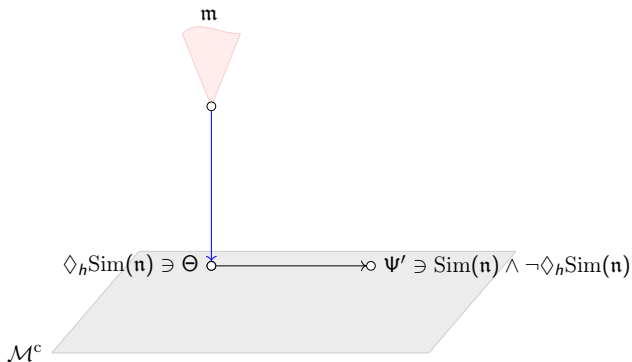
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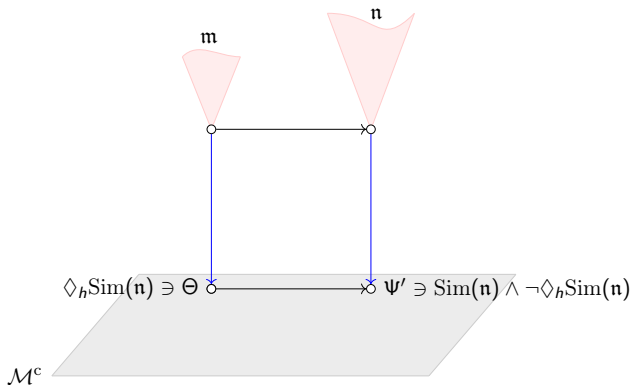
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Completeness of $[GL, GL]^e$

Theorem

$[GL, GL]^e$ is sound and complete wrt $(GL \times GL)^e$ and $GL \times GL$.

Proof.

It suffices to show that the sequence in quasimodel extraction stabilizes.

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- Since embedding preserves root and labeling,

$$\text{Sim}(m_i) \in \Psi_j \implies \Diamond_h \text{Sim}(m_i) \in \Psi_i \quad !!!$$

contradicting the construction $\neg \Diamond_h \text{Sim}(m_i) \in \Psi_i$.

Concluding remarks

- We found a partial answer to the 20y-open question in [Gabelaia et al. 06] for e-logics with GL as the vertical component.

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- We found a partial answer to the 20y-open question in [Gabelaia et al. 06] for e-logics with GL as the vertical component.

Q1. What about the other decidable logics?

Q2. For the incomplete commutators, is the e-logic still finitely axiomatisable?

Q3. Investigate intuitionistic companions.

E.g. $[GL, GL]^e$ provides a modal companion for a decidable (but proper) extension of IGL [Das et al. 2024].

$\Rightarrow$ can we also axiomatise a modal companion for IGL?

Thanks for your attention!

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want details? see

