Münchhausen provability and applications

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Gödel Löb Logic

Definition

The logic GL is a propositional logic with one modality $\Box$ given by the following axioms:

1. all propositional tautologies,
2. Distributivity: $\Box(\varphi \to \psi) \to (\Box\varphi \to \Box\psi)$,
3. Transitivity: $\Box\varphi \to \Box\Box\varphi$,
4. Löb: $\Box(\Box\varphi \to \varphi) \to \Box\varphi$.

The rules are Modus Ponens and Necessitation: $\dfrac{\varphi}{\Box\varphi}$.

Theorem

$\mathsf{GL} \vdash \varphi \quad \Leftrightarrow \quad \forall^* \mathrm{PA} \vdash \varphi^*$

Polymodal provability logic, transfinite

Definition

For Λ an ordinal or the class of all ordinals, the logic GLP_Λ is given by the following axioms:

1. all propositional tautologies,
2. Distributivity: $[\xi](\varphi \to \psi) \to ([\xi]\varphi \to [\xi]\psi)$ for all $\xi < \Lambda$,
3. Transitivity: $[\xi]\varphi \to [\xi][\xi]\varphi$ for all $\xi < \Lambda$,
4. Löb: $[\xi]([\xi]\varphi \to \varphi) \to [\xi]\varphi$ for all $\xi < \Lambda$,
5. Monotonicity: $[\xi]\varphi \to [\zeta]\varphi$ for $\xi < \zeta < \Lambda$,
6. Negative introspection: $\langle\xi\rangle\varphi \to [\zeta]\langle\xi\rangle\varphi$ for $\xi < \zeta < \Lambda$.

The rules are Modus Ponens and Necessitation for each modality: $\dfrac{\varphi}{[\xi]\varphi}$.

One-Münchhausen provability

Definition (Münchhausen theory and predicate)

Let T be a theory and let Λ denote an ordinal equipped with a representation in the language of T with corresponding represented ordering $\prec$. For this representation, it is required that

$$T \vdash \text{“}{\prec} \text{ is transitive, right-discrete and has a minimal element''},$$
$$T \vdash (\xi \prec \zeta) \to [\zeta]^{\Lambda}_{T}(\xi \prec \zeta),$$
$$T \vdash \neg(\xi \prec \zeta) \to [\zeta]^{\Lambda}_{T}\neg(\xi \prec \zeta),$$
$$\xi < \zeta < \Lambda \text{ implies } T \vdash \xi \prec \zeta.$$

We call T a Λ-*One-Münchhausen Theory* whenever there is a binary predicate $[\xi]^{\Lambda}_{T}\varphi$ with free variables ξ and φ so that

$$T \vdash \forall \phi\, \forall \zeta{\prec}\Lambda \Big([\zeta]^{\Lambda}_{T}\phi \leftrightarrow \Box_T \phi \vee \exists \psi\, \exists\, \xi{\prec}\zeta\, \big(\langle \xi \rangle^{\Lambda}_{T}\psi \wedge \Box_T(\langle \xi \rangle^{\Lambda}_{T}\psi \to \phi)\big) \Big)$$

Friedman-Goldfarb-Harrington

Theorem (FGH theorem)

Let T be any computably enumerable theory extending EA. For each $\sigma \in \Sigma_1$ we have that there is some $\rho \in \Sigma_1$ so that

$$\mathrm{EA} \vdash \Diamond_T \top \to \ (\sigma \leftrightarrow \Box_T \rho).$$

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$$\mathrm{I}\Sigma_n \vdash \langle n \rangle^{\Pi}_{T} \top \to \ \big(\sigma \leftrightarrow [n]^{\Pi}_{T}\rho\big).$$
- For each $\sigma(x) \in \Sigma_{n+1}$ we have that there is some $\rho(x) \in \Sigma_{n+1}$ so that
$$\mathrm{I}\Sigma_n \vdash \langle n \rangle^{\Pi}_{T} \top \to \ \big(\sigma(x) \leftrightarrow [n]^{\Pi}_{T}\rho(\dot{x})\big).$$

FGH and finite Turing jumps

Corollary

Let T be any sound c.e. theory and let $A \subseteq \mathbb{N}$. The following are equivalent

- *A is c.e. in $\emptyset^{(n)}$;*

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- *A is definable on the standard model by a formula of the form $[n]^{\Pi}_{T}\rho(\dot{x})$;*

Provability recursions

$$
\begin{aligned}
[0]^{\Box}_{T}\phi &:= \Box_T\phi, \quad \text{and} \\
[n+1]^{\Box}_{T}\phi &:= \Box_T\phi \ \vee \ \exists\,\psi \bigvee_{0\leq m\leq n} \Big(\langle m\rangle^{\Box}_{T}\psi \ \wedge \ \Box(\langle m\rangle^{\Box}_{T}\psi \to \phi)\Big).
\end{aligned}
\tag{1}
$$

Provability and finite Turing jumps

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One-Münchhausen provability

$$T \vdash \forall\phi\,\forall\zeta{\prec}\Lambda\Big(\,[\zeta]^{\Lambda}_{T}\phi \;\leftrightarrow\; \Box_T\phi \vee \exists\psi\,\exists\,\xi{\prec}\zeta\;\big(\langle\xi\rangle^{\Lambda}_{T}\psi \wedge \Box_T(\,\langle\xi\rangle^{\Lambda}_{T}\psi \to \phi)\big)\,\Big)$$

Soundness for GLP

Theorem (GLP Soundness for Munchhausen)

Let T be a Λ-One-Münchhausen theory and let $[\alpha]^{\Lambda}_{T}$ be a corresponding provability predicate.

If T proves transfinite $\Pi^0_1([\alpha]^{\Lambda}_{T})$ induction along Λ we have that T proves that all the rules and axioms of GLP_{Λ} are sound wr.t. T by interpreting $[\alpha]$ as $[\alpha]^{\Lambda}_{T}$.

Weakening the base theory

$$[\alpha]^{\boxtimes}_T\varphi \leftrightarrow \Box_T\varphi \vee \exists\sigma\,\exists\tau\,\Big(\,|\sigma| = |\tau| \wedge \forall\, i{<}|\tau|\; \tau_i{\prec}\alpha \wedge \forall\, i{<}|\sigma|\; \langle\tau_i\rangle^{\boxtimes}_T\sigma(i) \\ \wedge\; \Box_T\big(\forall\, i{<}|\sigma|\; \langle\tau_i\rangle^{\boxtimes}_T\sigma(i) \to \varphi\big)\Big). \quad (2)$$

Theorem

Let T be a Λ-Münchhausen theory and let $[\alpha]^{\boxtimes}_T$ be a corresponding Münchhausen provability predicate. Then, GLP_Λ is sound for T when the $[\alpha]$ -modalities ($\alpha \prec \Lambda$) are interpreted as $[\alpha]^{\boxtimes}_T$.

Complete for Turing jumps

- Transfinite Turing jumps can be related to Münchausen provability

Theorem

Given a well-behaved primitive recursive ordinal notation system for some limit ordinal $\langle \Lambda, \prec \rangle$*, let* T *be a sound theory proving* (2)*. For each* $\alpha \prec \Lambda$ *there is a formula* $\psi_\alpha(x)$ *so that*

$$x \in \varnothing^{(1+\alpha)} \iff \mathbb{N} \models [\alpha]_T^{\boxtimes} \psi_\alpha(\overline{x}).$$

Moreover, ψ_α *can be obtained by primitive recursion from* α*.*

Proof ingredients

Lemma

Let T be a Münchhausen theory with corresponding provability predicate $[\xi]\theta$ and let $U \supseteq T$ so that $U \vdash \mathrm{B}\Sigma_1([\alpha]\varphi)$. We then have

$$U \vdash \forall\beta\,\forall\varphi\,\exists\rho\,\Big(\langle\beta+1\rangle\top \to\ \Big[\exists x\langle\beta\rangle\varphi(\dot{x})\ \longleftrightarrow\ [\beta+1]\rho\Big]\Big),$$

More proof ingredients

Lemma

There is a computable function g so that for $\alpha, \lambda \prec \Lambda$ and λ a limit ordinal we have

1. $$x \in \varnothing^{(1+\alpha+1)} \iff \exists s\, g(s, x) \notin \varnothing^{(1+\alpha)};$$
2. *Something for limits.*

Combining: the successor case

$$
\begin{array}{rcll}
x \in \varnothing^{1+\alpha+1} & \Leftrightarrow & \exists s\, g(s,x) \notin \varnothing^{(1+\alpha)} & \\
 & \Leftrightarrow & \exists s\, \neg\big(g(s,x) \in \varnothing^{(1+\alpha)}\big) & \\
 & \Leftrightarrow & \exists s\, \neg[\alpha]\psi_\alpha\big(g(\dot{s},\dot{x})\big) & \text{(by the IH)} \\
 & \Leftrightarrow & \exists s\, \langle\alpha\rangle\neg\psi_\alpha\big(g(\dot{s},\dot{x})\big) & \\
 & \Leftrightarrow & [\alpha+1]\rho(\dot{x}) &
\end{array}
$$

Fundamental sequences

Lemma

Let λ be a limit ordinal with fixed fundamental sequence $\{\lambda[\![x]\!]\}_{x\in\omega}$. Moreover, let T be a Münchhausen theory with corresponding provability predicate $[\xi]\theta$ and let $U \supseteq T$ so that $U \vdash \mathrm{B}\Sigma_1([\alpha]\varphi)$. We then have

$$U \vdash \forall\varphi\,\exists\rho\,\Big(\langle\lambda\rangle\top \to \Big[\exists x\langle\lambda[\![x]\!]\rangle\varphi(\dot{x}) \;\longleftrightarrow\; [\lambda]\rho(\dot{x})\Big]\Big).$$

Moreover, ρ can be obtained from λ and φ in a primitive recursive way.

More proof ingredients

Lemma

There are a computable functions g, h so that for $\alpha, \lambda \prec \Lambda$ and λ a limit ordinal we have

1. *There is a computable function g so that*

$$x \in \varnothing^{(1+\alpha+1)} \iff \exists s\, g(s,x) \notin \varnothing^{(1+\alpha)};$$

2.

$$x \in \varnothing^{(\lambda)} \iff \exists s\, h(s,x) \notin \varnothing^{(1+\lambda[\![s]\!])}.$$

Combining: the limit case

$$
\begin{array}{rcl}
x \in \varnothing^{\lambda} & \Leftrightarrow & \exists s\, h(s,x) \notin \varnothing^{(1+\lambda[\![s]\!])} \\
& \Leftrightarrow & \exists s\, \neg\big(h(s,x) \in \varnothing^{(1+\lambda[\![s]\!])}\big) \\
& \Leftrightarrow & \exists s\, \neg[\lambda[\![s]\!]]\psi_{\lambda[\![s]\!]}\big(h(\dot{s},\dot{x})\big) \qquad \text{(by the IH)} \\
& \Leftrightarrow & \exists s\, \langle\lambda[\![s]\!]\rangle\neg\psi_{\lambda[\![s]\!]}\big(h(\dot{s},\dot{x})\big) \\
& \Leftrightarrow & [\lambda]\rho(\dot{x})
\end{array}
$$

Wrapping up

Theorem

...

$$x \in \varnothing^{(1+\alpha)} \iff \mathbb{N} \models [\alpha]^{\boxtimes}_{T}\psi_\alpha(\overline{x}).$$

using our earlier results and

Lemma (Computable Recursion Theorem)

Let $\langle \Lambda, \prec \rangle$ be a primitive recursive ordinal notation system. For every combination of primitive recursions b, g and h of the right arities there is a unique primitive recursion f that satisfies the following equations:

$$\begin{aligned} &f(0,x) = b(x);\\ &f(\alpha+1,x) = g(\alpha,x,f(\alpha,x));\\ &f(\lambda,x) = h(\{f(\alpha,x) \mid \alpha \prec \lambda\},x) \quad \textit{for limit } \lambda. \end{aligned}$$

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- Joosten, J.J.: Münchhausen provability. Journal of Symbolic Logic **86**(3), 1006–1034 (2021).