Feferman Interpretability and Applications
Logic Colloquium 2024, Gothenburg, Sweden

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- A theory U *interprets* a theory V when there is a *translation* j
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- The translation j preserves logical structure,
  for example
  $$(\varphi \vee \psi)^j := \varphi^j \vee \psi^j$$
  etc.
  In particular
  $$\bot^j := \bot.$$

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- Each theory T with an RE set of axioms is deductively equivalent to a theory T' with a REC set of axioms
  - Proof: with Axioms(T) $= \{t_1, t_2, t_3 \ldots\}$ and $f(n) = t_n$
    let Axioms(T') $:= \{t'_1, t'_2, t'_3 \ldots\}$ be given by

$$t'_1 := t_1;$$
$$t'_2 := t_2 \wedge t_2;$$
$$t'_3 := t_3 \wedge t_3 \wedge t_3;$$
$$\ldots$$

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...

- All considered theories are poly-time axiomatized.

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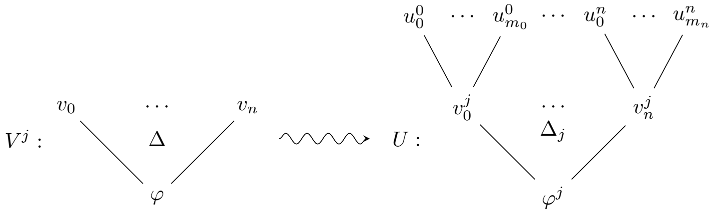
- In general

$$T \nvdash (U \rhd V) \;\rightarrow\; \Box_U (V \equiv V')$$

Feferman's trick

- $\mathsf{Axiom}_{V^j}(\nu) \;:\longleftrightarrow\; \big(\mathsf{Axiom}_V(\nu) \wedge \Box_U \nu^j\big)$



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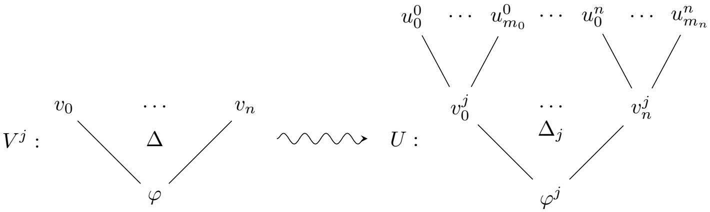

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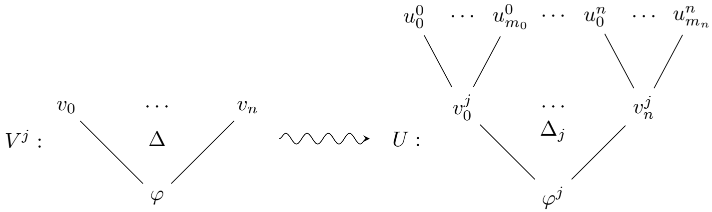
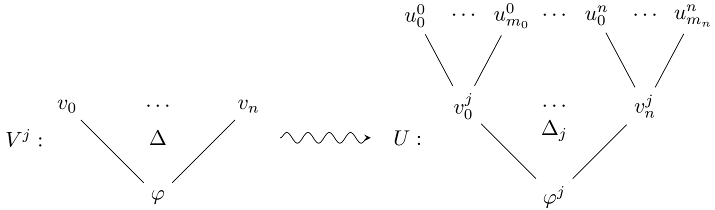


- $U \rhd V^j$ becomes a mere triviality
- Works for strong enough theories

Troubles of the weak



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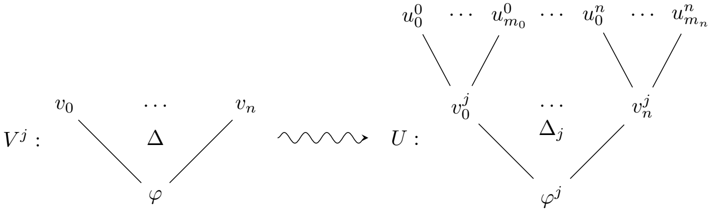

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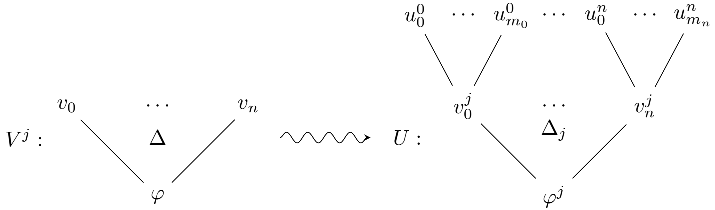

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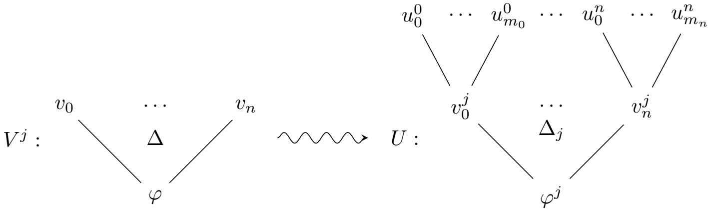

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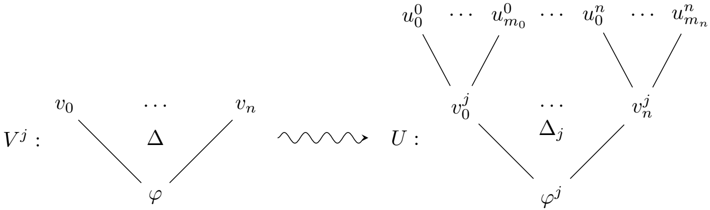

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  - totality of exp needed for the Gödel numbers to exist

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$\exists\, v, p{<}\nu \Big(\nu = \ulcorner v \wedge (\overline{p} = \overline{p})\urcorner \wedge \mathsf{Axiom}_V(\nu) \wedge \mathsf{Proof}_U(p, v^j)\Big)$

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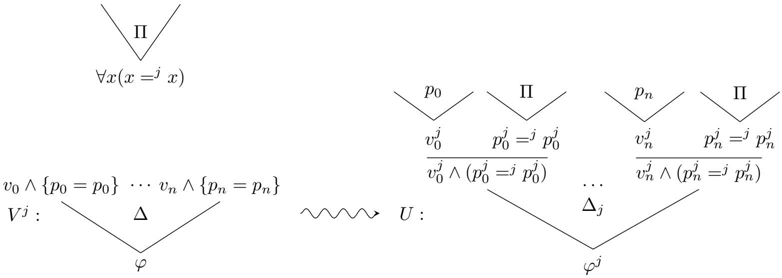

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- In general
  $$\mathsf{S}^1_2 \nvdash j : (U \rhd V) \;\rightarrow\; \quad \Box_U(V \equiv V^j)$$

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- Just as the modal expression

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$$\Box_T A^*.$$

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Some obvious principles define the basic logic **IL**

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  - $\Big((A \rhd B) \wedge (B \rhd C)\Big) \to (A \rhd C)$;
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  - $\Big((A \rhd B) \land (B \rhd C)\Big) \to (A \rhd C)$;
  - $\Big((A \rhd C) \land (B \rhd C)\Big) \to \Big((A \lor B) \rhd C\Big)$;
  - $(A \rhd B) \to (\Diamond A \to \Diamond B)$;
  - $\Diamond A \rhd A$.
- The only rules are Modus Ponens and Necessitation: $\frac{A}{\Box A}$.

Feferman and Visser in action

IL axioms:

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$\Box A \to \Box\Box A$
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$(A \rhd B) \to (\Diamond A \to \Diamond B)$
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$A \rhd B \to \Box(A \rhd^k B)$

More details and applications at

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$\big((A \rhd B) \wedge (B \rhd C)\big) \to (A \rhd C)$
$\big((A \rhd C) \wedge (B \rhd C)\big) \to \big((A \vee B) \rhd C\big)$
$(A \rhd B) \to (\Diamond A \to \Diamond B)$
$\Diamond A \rhd A.$

$$A \rhd B \to \Box(A \rhd^k B)$$
$$\to \Box(\Diamond A \to \Diamond^k B)$$
$$\to \Box(\Diamond A \wedge \Box C \to \Diamond^k B \wedge \Box C)$$
$$\to \Box(\Diamond A \wedge \Box C \to \Diamond^k B \wedge \Box^k \Box C)$$
$$\to \Box\Big(\Diamond A \wedge \Box C \to \Diamond^k (B \wedge \Box C)\Big)$$

More details and applications at

https://arxiv.org/abs/2406.18506

Feferman and Visser in action

IL axioms:

$\Box(A \to B) \to (\Box A \to \Box B)$

$\Box A \to \Box\Box A$

$\Box(\Box A \to A) \to \Box A$

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$\big((A \rhd B) \wedge (B \rhd C)\big) \to (A \rhd C)$

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$$\to \Box\Big(\Diamond A \wedge \Box C \to \Diamond^k (B \wedge \Box C)\Big)$$
$$\to (\Diamond A \wedge \Box C) \rhd \Diamond^k (B \wedge \Box C)$$

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Thanks for feedback!

Workshop on Proof Theory, Modal Logic and Reflection Principles

Invited Speakers:

- Michael Rathjen
- Gilda Ferreira
- Juan Aguilera
- Alex Kavvos
- Daniyar Shamkanov
- Raheleh Jalali

Tutorials:

- Mateusz Łełyk
- Ali Enayat

Important Dates:

- Conference: **September 2–5, Ghent University**
- Paper Submission Deadline: **July 15**

Website: `wormshop2024.ugent.be`