Transfinite Turing Jumps through Provability

Computability in Europe 2024

Amsterdam

Joost J. Joosten

University of Barcelona

Monday 08-07-2024

From finite to transfinite

Turing jumps through provability

Joost J. Joosten

University of Barcelona

Thursday 02-07-2015
Computability in Europe, Bucharest

Post's Theorem

- Let $A \subseteq \mathbb{N}$; Post's Theorem says that the following are equivalent:

Post's Theorem

- Let $A \subseteq \mathbb{N}$; Post's Theorem says that the following are equivalent:
  - A is c.e. in $\emptyset^{(n)}$ iff;

Post's Theorem

- Let $A \subseteq \mathbb{N}$; Post's Theorem says that the following are equivalent:
  - A is c.e. in $\emptyset^{(n)}$ iff;
  - A is many-one reducible to $\emptyset^{(n+1)}$ iff;

Post's Theorem

- Let $A \subseteq \mathbb{N}$; Post's Theorem says that the following are equivalent:
  - A is c.e. in $\emptyset^{(n)}$ iff;
  - A is many-one reducible to $\emptyset^{(n+1)}$ iff;
  - A is definable on the standard model by a Σ_{n+1} formula.

Post's Theorem

- Let $A \subseteq \mathbb{N}$; Post's Theorem says that the following are equivalent:
  - A is c.e. in $\emptyset^{(n)}$ iff;
  - A is many-one reducible to $\emptyset^{(n+1)}$ iff;
  - A is definable on the standard model by a Σ_{n+1} formula.
- This can be seen as

Post's Theorem

- Let $A \subseteq \mathbb{N}$; Post's Theorem says that the following are equivalent:
  - A is c.e. in $\emptyset^{(n)}$ iff;
  - A is many-one reducible to $\emptyset^{(n+1)}$ iff;
  - A is definable on the standard model by a Σ_{n+1} formula.
- This can be seen as

  Turing jumps through syntax

Post's Theorem

- Let $A \subseteq \mathbb{N}$; Post's Theorem says that the following are equivalent:
  - A is c.e. in $\emptyset^{(n)}$ iff;
  - A is many-one reducible to $\emptyset^{(n+1)}$ iff;
  - A is definable on the standard model by a Σ_{n+1} formula.
- This can be seen as

  Turing jumps through syntax
- How to generalize to transfinite Turing jumps?

Post's Theorem

- Let $A \subseteq \mathbb{N}$; Post's Theorem says that the following are equivalent:
  - A is c.e. in $\emptyset^{(n)}$ iff;
  - A is many-one reducible to $\emptyset^{(n+1)}$ iff;
  - A is definable on the standard model by a Σ_{n+1} formula.
- This can be seen as

  Turing jumps through syntax

- How to generalize to transfinite Turing jumps?
- Ingredients to our solution:

Post's Theorem

- Let $A \subseteq \mathbb{N}$; Post's Theorem says that the following are equivalent:
  - A is c.e. in $\emptyset^{(n)}$ iff;
  - A is many-one reducible to $\emptyset^{(n+1)}$ iff;
  - A is definable on the standard model by a Σ_{n+1} formula.
- This can be seen as

  Turing jumps through syntax
- How to generalize to transfinite Turing jumps?
- Ingredients to our solution:
  - Provability;

Post's Theorem

- Let $A \subseteq \mathbb{N}$; Post's Theorem says that the following are equivalent:
  - A is c.e. in $\emptyset^{(n)}$ iff;
  - A is many-one reducible to $\emptyset^{(n+1)}$ iff;
  - A is definable on the standard model by a Σ_{n+1} formula.
- This can be seen as

  Turing jumps through syntax

- How to generalize to transfinite Turing jumps?
- Ingredients to our solution:
  - Provability;
  - Friedman-Goldfarb-Harrington Theorem and generalizations;

Post's Theorem

- Let $A \subseteq \mathbb{N}$; Post's Theorem says that the following are equivalent:
  - A is c.e. in $\emptyset^{(n)}$ iff;
  - A is many-one reducible to $\emptyset^{(n+1)}$ iff;
  - A is definable on the standard model by a Σ_{n+1} formula.
- This can be seen as

  Turing jumps through syntax

- How to generalize to transfinite Turing jumps?
- Ingredients to our solution:
  - Provability;
  - Friedman-Goldfarb-Harrington Theorem and generalizations;
  - Recursively apply the FGH theorem to eliminate auxiliary syntactical notions.

Friedman-Goldfarb-Harrington

Theorem (FGH theorem)

Let T be any computably enumerable theory extending EA. For each $\sigma \in \Sigma_1$ we have that there is some $\rho \in \Sigma_1$ so that

$$\mathrm{EA} \vdash \Diamond_T \top \to \ \big(\sigma \leftrightarrow \Box_T \rho\big).$$

Generalizing FGH

- Generalizations of the FGH theorem:

Generalizing FGH

- Generalizations of the FGH theorem:
- Let $[n]^{\Pi}_{T}\rho$ denote a natural formalization of "*ρ is provable in T together with all true Π^0_n sentences*".

Generalizing FGH

- Generalizations of the FGH theorem:
- Let $[n]^{\Pi}_{T}\rho$ denote a natural formalization of "*ρ is provable in T together with all true Π^0_n sentences*".
- For each $\sigma \in \Sigma_{n+1}$ we have that there is some $\rho \in \Sigma_{n+1}$ so that

$$\mathrm{I}\Sigma_n \vdash \langle n\rangle^{\Pi}_{T}\top \to \ \big(\sigma \leftrightarrow [n]^{\Pi}_{T}\rho\big).$$

Generalizing FGH

- Generalizations of the FGH theorem:
- Let $[n]^{\Pi}_{T}\rho$ denote a natural formalization of "*ρ is provable in T together with all true Π^0_n sentences*".
- For each $\sigma \in \Sigma_{n+1}$ we have that there is some $\rho \in \Sigma_{n+1}$ so that

$$\mathrm{I}\Sigma_n \vdash \langle n\rangle^{\Pi}_{T}\top \to \ \big(\sigma \leftrightarrow [n]^{\Pi}_{T}\rho\big).$$

- For each $\sigma(x) \in \Sigma_{n+1}$ we have that there is some $\rho(x) \in \Sigma_{n+1}$ so that

$$\mathrm{I}\Sigma_n \vdash \langle n\rangle^{\Pi}_{T}\top \to \ \big(\sigma(x) \leftrightarrow [n]^{\Pi}_{T}\rho(\dot{x})\big).$$

FGH and finite Turing jumps

Corollary

Let T be any sound c.e. theory and let $A \subseteq \mathbb{N}$. The following are equivalent

- *A is c.e. in $\emptyset^{(n)}$;*

FGH and finite Turing jumps

Corollary

Let T be any sound c.e. theory and let $A \subseteq \mathbb{N}$. The following are equivalent

- *A is c.e. in $\emptyset^{(n)}$;*
- *A is many-one reducible to $\emptyset^{(n+1)}$;*

FGH and finite Turing jumps

Corollary

Let T be any sound c.e. theory and let $A \subseteq \mathbb{N}$. The following are equivalent

- *A is c.e. in $\emptyset^{(n)}$;*
- *A is many-one reducible to $\emptyset^{(n+1)}$;*
- *A is definable on the standard model by a Σ_{n+1} formula;*

FGH and finite Turing jumps

Corollary

Let T be any sound c.e. theory and let $A \subseteq \mathbb{N}$. The following are equivalent

- *A is c.e. in $\emptyset^{(n)}$;*
- *A is many-one reducible to $\emptyset^{(n+1)}$;*
- *A is definable on the standard model by a Σ_{n+1} formula;*
- *A is definable on the standard model by a formula of the form $[n]_T^{\Pi}\rho(\dot{x})$;*

Provability recursions

$$
\begin{aligned}
[0]^{\Box}_{T}\phi \quad &:= \Box_T\phi, \quad \text{and} \\
[n+1]^{\Box}_{T}\phi \quad &:= \ \Box_T\phi \ \vee \ \exists \psi \bigvee_{0\leq m\leq n} \Big(\langle m\rangle^{\Box}_{T}\psi \ \wedge \ \Box(\langle m\rangle^{\Box}_{T}\psi \to \phi)\Big).
\end{aligned} \tag{1}
$$

Provability and finite Turing jumps

Corollary

Let T be any sound c.e. theory and let $A \subseteq \mathbb{N}$. The following are equivalent

- *A is c.e. in $\emptyset^{(n)}$;*
- *A is many-one reducible to $\emptyset^{(n+1)}$;*
- *A is definable on the standard model by a Σ_{n+1} formula;*
- *A is definable on the standard model by a formula of the form $[n+1]^{\Box}_{T}\rho(\dot{x})$;*

Provability recursions

$$
\begin{aligned}
[0]^{\Box}_T\phi \quad &:= \Box_T\phi, \quad \text{and} \\
[n+1]^{\Box}_T\phi \quad &:= \ \Box_T\phi \ \vee \ \exists\,\psi \bigvee_{0\leq m\leq n} \Big(\langle m\rangle^{\Box}_T\psi \ \wedge \ \Box(\langle m\rangle^{\Box}_T\psi \to \phi)\Big).
\end{aligned}
\tag{2}
$$

Main idea:

$$
[\xi]^{\Box}_T\varphi \ :\leftrightarrow \ \Box_T\varphi \vee \exists\,\zeta{<}\xi\,\exists\psi\ (\langle\zeta\rangle^{\Box}_T\psi \ \wedge \ \Box_T(\langle\zeta\rangle^{\Box}_T\psi \to \varphi))
$$

A schematic approach

Definition (Münchhausen theory and predicate)

Let T be a theory and let Λ denote an ordinal equipped with a representation in the language of T with corresponding represented ordering $\prec$. For this representation, it is required that

$$
\begin{array}{l}
T \vdash \text{“}\prec \text{ is transitive, right-discrete and has a minimal element”}, \\
T \vdash (\xi \prec \zeta) \to [\zeta]^{\Lambda}_{T}(\xi \prec \zeta), \\
T \vdash \neg(\xi \prec \zeta) \to [\zeta]^{\Lambda}_{T}\neg(\xi \prec \zeta), \\
\xi < \zeta < \Lambda \text{ implies } T \vdash \xi \prec \zeta.
\end{array}
$$

We call T a Λ-*One-Münchhausen Theory* whenever there is a binary predicate $[\xi]^{\Lambda}_{T}\varphi$ with free variables ξ and φ so that

$$
T \vdash \forall \phi \, \forall \zeta {\prec} \Lambda \Big([\zeta]^{\Lambda}_{T}\phi \leftrightarrow \Box_T \phi \vee \exists \psi \, \exists \, \xi {\prec} \zeta \, \big(\langle \xi \rangle^{\Lambda}_{T}\psi \wedge \Box_T (\langle \xi \rangle^{\Lambda}_{T}\psi \to \phi) \big) \Big)
$$

Polymodal provability logic, transfinite

Definition

For Λ an ordinal or the class of all ordinals, the logic GLP_Λ is given by the following axioms:

1. all propositional tautologies,
2. Distributivity: $[\xi](\varphi \to \psi) \to ([\xi]\varphi \to [\xi]\psi)$ for all $\xi < \Lambda$,
3. Transitivity: $[\xi]\varphi \to [\xi][\xi]\varphi$ for all $\xi < \Lambda$,
4. Löb: $[\xi]([\xi]\varphi \to \varphi) \to [\xi]\varphi$ for all $\xi < \Lambda$,
5. Monotonicity: $[\xi]\varphi \to [\zeta]\varphi$ for $\xi < \zeta < \Lambda$,
6. Negative introspection: $\langle\xi\rangle\varphi \to [\zeta]\langle\xi\rangle\varphi$ for $\xi < \zeta < \Lambda$.

The rules are Modus Ponens and Necessitation for each modality: $\dfrac{\varphi}{[\xi]\varphi}$.

Soundness for GLP

Theorem (GLP Soundness for Munchhausen)

Let T be a Λ-One-Münchhausen theory and let $[\alpha]^{\Lambda}_{T}$ be a corresponding provability predicate.

If T proves transfinite $\Pi^0_1([\alpha]^{\Lambda}_{T})$ induction along Λ we have that T proves that all the rules and axioms of GLP_{Λ} are sound wr.t. T by interpreting $[\alpha]$ as $[\alpha]^{\Lambda}_{T}$.

Weakening the base theory

$$[\alpha]^{\boxtimes}_T\varphi \leftrightarrow \Box_T\varphi \vee \exists\sigma\,\exists\tau\,\Big(\,|\sigma|=|\tau| \wedge \forall\, i{<}|\tau|\ \tau_i{\prec}\alpha \wedge \forall\, i{<}|\sigma|\ \langle\tau_i\rangle^{\boxtimes}_T\sigma(i) \wedge \Box_T\big(\forall\, i{<}|\sigma|\ \langle\tau_i\rangle^{\boxtimes}_T\sigma(i) \rightarrow \varphi\big)\Big). \quad (3)$$

Theorem

Let T be a Λ-Münchhausen theory and let $[\alpha]^{\boxtimes}_T$ be a corresponding Münchhausen provability predicate. Then, GLP_Λ is sound for T when the $[\alpha]$ -modalities ($\alpha \prec \Lambda$) are interpreted as $[\alpha]^{\boxtimes}_T$.

Complete for Turing jumps

- Transfinite Turing jumps can be related to Münchausen provability

Theorem

Given a well-behaved primitive recursive ordinal notation system for some limit ordinal $\langle \Lambda, \prec \rangle$, let T be a sound theory proving (3). For each $\alpha \prec \Lambda$ there is a formula $\psi_\alpha(x)$ so that

$$x \in \varnothing^{(1+\alpha)} \iff \mathbb{N} \models [\alpha]_T^{\boxtimes} \psi_\alpha(\overline{x}).$$

Moreover, ψ_α can be obtained by primitive recursion from α.

Proof ingredients

Lemma

Let T be a Münchhausen theory with corresponding provability predicate $[\xi]\theta$ and let $U \supseteq T$ so that $U \vdash \mathrm{B}\Sigma_1([\alpha]\varphi)$. We then have

$$U \vdash \forall \beta \, \forall \varphi \, \exists \rho \, \Big(\langle \beta + 1 \rangle \top \to \ \Big[\exists x \langle \beta \rangle \varphi(\dot{x}) \ \longleftrightarrow \ [\beta + 1]\rho \Big] \Big),$$

More proof ingredients

Lemma

There is a computable function g so that for $\alpha, \lambda \prec \Lambda$ and λ a limit ordinal we have

1. $$x \in \varnothing^{(1+\alpha+1)} \iff \exists s\, g(s,x) \notin \varnothing^{(1+\alpha)};$$
2. *Something for limits.*

Combining: the successor case

$$\begin{array}{rcll} x \in \varnothing^{1+\alpha+1} & \Leftrightarrow & \exists s\, g(s,x) \notin \varnothing^{(1+\alpha)} & \\ & \Leftrightarrow & \exists s\, \neg\big(g(s,x) \in \varnothing^{(1+\alpha)}\big) & \\ & \Leftrightarrow & \exists s\, \neg[\alpha]\psi_\alpha\big(g(\dot{s},\dot{x})\big) & \text{(by the IH)} \\ & \Leftrightarrow & \exists s\, \langle\alpha\rangle\neg\psi_\alpha\big(g(\dot{s},\dot{x})\big) & \\ & \Leftrightarrow & [\alpha+1]\rho(\dot{x}) & \end{array}$$

Fundamental sequences

Lemma

Let λ be a limit ordinal with fixed fundamental sequence $\{\lambda[\![x]\!]\}_{x\in\omega}$. Moreover, let T be a Münchhausen theory with corresponding provability predicate $[\xi]\theta$ and let $U \supseteq T$ so that $U \vdash \mathrm{B}\Sigma_1([\alpha]\varphi)$. We then have

$$U \vdash \forall\varphi\,\exists\rho\,\Big(\langle\lambda\rangle\top \to \Big[\exists x\langle\lambda[\![x]\!]\rangle\varphi(\dot{x}) \;\longleftrightarrow\; [\lambda]\rho(\dot{x})\Big]\Big).$$

Moreover, ρ can be obtained from λ and φ in a primitive recursive way.

More proof ingredients

Lemma

There are a computable functions g, h so that for $\alpha, \lambda \prec \Lambda$ and λ a limit ordinal we have

1. *There is a computable function g so that*

$$x \in \varnothing^{(1+\alpha+1)} \iff \exists s\, g(s,x) \notin \varnothing^{(1+\alpha)};$$

2. $$x \in \varnothing^{(\lambda)} \iff \exists s\, h(s,x) \notin \varnothing^{(1+\lambda[\![s]\!])}.$$

Combining: the successor case

$$
\begin{aligned}
x \in \varnothing^{\lambda} & \Leftrightarrow & \exists s\, h(s,x) \notin \varnothing^{(1+\lambda[\![s]\!])} \\
& \Leftrightarrow & \exists s\, \neg\big(h(s,x) \in \varnothing^{(1+\lambda[\![s]\!])}\big) \\
& \Leftrightarrow & \exists s\, \neg[\lambda[\![s]\!]]\psi_{\lambda[\![s]\!]}\big(h(\dot{s},\dot{x})\big) \quad \text{(by the IH)} \\
& \Leftrightarrow & \exists s\, \langle\lambda[\![s]\!]\rangle\neg\psi_{\lambda[\![s]\!]}\big(h(\dot{s},\dot{x})\big) \\
& \Leftrightarrow & [\lambda]\rho(\dot{x})
\end{aligned}
$$

Wrapping up

Theorem

...

$$x \in \varnothing^{(1+\alpha)} \iff \mathbb{N} \models [\alpha]_T^{\boxtimes}\psi_\alpha(\overline{x}).$$

using our earlier results and

Lemma (Computable Recursion Theorem)

Let $\langle \Lambda, \prec \rangle$ be a primitive recursive ordinal notation system. For every combination of primitive recursions b, g and h of the right arities there is a unique primitive recursion f that satisfies the following equations:

$$\begin{aligned} f(0,x) &= b(x); \\ f(\alpha+1,x) &= g(\alpha,x,f(\alpha,x)); \\ f(\lambda,x) &= h(\{f(\alpha,x) \mid \alpha \prec \lambda\},x) \quad \text{for limit } \lambda. \end{aligned}$$

Joosten, J.J.: Turing jumps through provability. In: Evolving Computability - 11th Conference on Computability in Europe, CiE 2015, Bucharest, Romania, June 29 - July 3, 2015. Proceedings. pp. 216–225 (2015).

Joosten, J.J.: Münchhausen provability. Journal of Symbolic Logic **86**(3), 1006–1034 (2021).