The Complexity of the Constructive Master Modality

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Motivation

Constructive modal logics are a kind of intuitionistic modal logics motivated by computational considerations.

- CK [Paiva01] $:=$ basic constructive logic with modal axioms $\Box(p \to q) \to (\Box p \to \Box q)$ and $\Box(p \to q) \to (\Diamond p \to \Diamond q)$.
- Wijesekera's logic WK [Wije90] $:=$ CK $+ \neg\Diamond\bot$.

In constructive logics:

- worlds $\to$ stages of computation,
- propositions $\to$ types describing code available/executable.

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In constructive logics:

- worlds $\to$ stages of computation,
- propositions $\to$ types describing code available/executable.

We enrich constructive logics with master modalities:

$\Box^* p \to$ in holds in all future computations and context extensions,
$\Diamond^* p \to$ the possibility of a future computation survives context extensions.

We define the constructive logics with the master modality and provide a chain of translations into PDL.

$$\begin{tikzcd}
\mathrm{PDL} & \supseteq & \mathrm{K}^* \arrow[r, hook] & \mathrm{CK}^*_\Box & \subseteq & \mathrm{CK}^* \arrow[r, hook] & \mathrm{WK}^* \arrow[r, hook] & \mathrm{PDL} \\
 & & & & & \mathrm{CS4} \arrow[u, hook] & \mathrm{WS4} \arrow[u, hook] &
\end{tikzcd}$$

Theorem ([Fischer79], [Pratt80])

PDL *is* EXPTIME*-complete and has the exponential-size model property.*

Remark. $\mathrm{CK}^*_\Box$ over the $\Diamond, \Diamond^*$-free fragment enjoys a Hilbert system [Celani01] and cyclic calculi [Afshari24].

Bi-relational semantics

Definition

A CK-model is a tuple $\mathcal{M} = \langle W, \preccurlyeq, R, \|\cdot\| \rangle$ where

- W is a set of worlds,
- $\preccurlyeq$ is a preorder,
- R is a relation on W and
- $\|\cdot\| : \mathrm{Prop} \cup \{\bot\} \to \mathcal{P}(W)$ is a valuation satisfying

  (i) $\|\bot\| \subseteq \|p\|$,

  (ii) if $w \preccurlyeq v \wedge w \in \|p\|$ then $v \in \|p\|$,

  (iii) if $w \in \|\bot\| \wedge (w \preccurlyeq v \vee wRv)$ then $v \in \|\bot\|$,

  (iv) if $w \in \|\bot\|$ then $\exists v.\ wRv$.

A WK-*model* is an infallible ($\|\bot\| = \varnothing$) CK-model.

Constructive logics with the master-modality

Definition

The language $\mathcal{L}^*$ is defined by

$$\varphi ::= \bot \mid p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \to \varphi \mid \Box\varphi \mid \Diamond\varphi \mid \Box^*\varphi \mid \Diamond^*\varphi.$$

$\mathcal{L}^*_\Box$ is the $\Diamond, \Diamond^*$-free fragment of $\mathcal{L}^*$.

Definition

$\mathsf{CK}^* :=$ formulas of $\mathcal{L}^*$ that are valid on all CK-models;
$\mathsf{WK}^* :=$ formulas of $\mathcal{L}^*$ that are valid on all WK-models;
$\mathsf{CK}^*_\Box :=$ formulas of $\mathcal{L}^*_\Box$ that are valid on all CK-models.

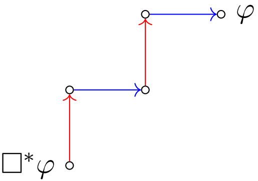
Proposition

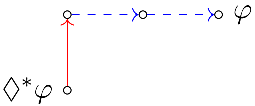
In the fragment $\mathcal{L}^*_\Box$, validity is independent of infallibility ($\mathsf{CK}^*_\Box$ and $\mathsf{WK}^*_\Box$ coincide).

Interpreting the master-modality

The satisfaction relation $(\mathcal{M}, w) \Vdash \varphi$ is defined as usual, with

- $(\mathcal{M}, w) \Vdash \Box^* \varphi$ iff $\forall v.\ w\ (\preccurlyeq; R)^*\ v \to (\mathcal{M}, v) \Vdash \varphi$;
- $(\mathcal{M}, w) \Vdash \Diamond^* \varphi$ iff $\forall v \succcurlyeq w.\ \exists u.\ vR^*u\ \wedge\ (\mathcal{M}, u) \Vdash \varphi$.





Lemma

$(\mathcal{M}, w) \Vdash \Box^* \varphi$ iff $\forall v.\ w(\preccurlyeq; R^*)^* v \to (\mathcal{M}, v) \Vdash \varphi$.

Translating CK* into WK*

The translation $\omega : \mathcal{L}^* \to \mathcal{L}^*$ is defined as

$$\omega(\varphi) := \varphi\big[\bot \leftarrow \Box^*(\bigwedge \mathrm{Var}(\varphi) \wedge p_\bot \wedge \Diamond p_\bot)\big]$$

for $p_\bot \notin Var(\varphi)$. $|\omega(\varphi)|$ is polynomial on $|\varphi|$.

Theorem

$\Vdash_{\mathsf{CK}^*} \varphi$ *iff* $\Vdash_{\mathsf{WK}^*} \omega(\varphi)$ *for any* $\varphi \in \mathcal{L}^*$.

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$\Vdash_{\mathsf{CK}^*} \varphi$ iff $\Vdash_{\mathsf{WK}^*} \omega(\varphi)$ for any $\varphi \in \mathcal{L}^*$.

Proof idea

($\Leftarrow$) : Transform a CK-model $\mathcal{M}^{\mathsf{CK}}$ into a WK-model $\mathcal{M}^{\mathsf{WK}}$ by updating the valuation to

$$\|p_\bot\|^{\mathsf{WK}} := \|\bot\|^{\mathsf{CK}} \quad \text{and} \quad \|\bot\|^{\mathsf{WK}} := \emptyset.$$

$\mathcal{M}^{\mathsf{CK}}$ satisfies a formula iff $\mathcal{M}^{\mathsf{WK}}$ satisfies its translation.

Definition

The language $\mathcal{L}^{\mathrm{PDL}}$ is defined as:

$$\varphi := p \mid \neg\varphi \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid [\alpha]\varphi$$
$$\alpha := a \mid (\alpha;\alpha) \mid \alpha^*$$

for $a \in \Pi_0$ a set of atomic programs. $\langle\alpha\rangle := \neg[\alpha]\neg$.

$\mathcal{L}^{\mathrm{K}^*}$ is the fragment of $\mathcal{L}^{\mathrm{PDL}}$ with only $[a] := \Box$ and $[a^*] := \Box^*$.

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Definition

A PDL-model is a triple $\mathcal{M} = \langle W, \rho, \|\cdot\|\rangle$ where $\rho : \Pi_0 \to 2^{W\times W}$ is an interpretation of atomic programs.

ρ is used to interpret modalities classically by extension to all programs: $\rho(\alpha;\beta) := \rho(\alpha);\rho(\beta)$ and $\rho(\alpha^*) := \big(\rho(\alpha)\big)^*$.

Translating WK^* into PDL

We fix $\Pi_0 = \{i, m\}$ as the atomic programs of PDL.

Definition

The translation $\tau : \mathcal{L}^* \to \mathcal{L}^{\mathsf{PDL}}$ is defined by

- $\tau(\bot) := p_\bot \wedge \neg p_\bot$ for (any) $p_\bot \in \mathrm{Prop}$;
- $\tau(p) := [i^*]p$;
- $\tau(\varphi \odot \psi) := \tau(\varphi) \odot \tau(\psi)$ for $\odot \in \{\wedge, \vee\}$;
- $\tau(\varphi \to \psi) := [i^*](\neg\tau(\varphi) \vee \tau(\psi))$;

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- $\tau(\varphi \to \psi) := [i^*](\neg\tau(\varphi) \vee \tau(\psi))$;
- $\tau(\Box\varphi) := [i^*; m]\tau(\varphi)$;
- $\tau(\Box^*\varphi) := [(i^*; m)^*]\tau(\varphi)$;
- $\tau(\Diamond\varphi) := [i^*]\langle m\rangle\tau(\varphi)$;
- $\tau(\Diamond^*\varphi) := [i^*]\langle m^*\rangle\tau(\varphi)$;

$|\tau(\varphi)|$ is linear on $|\varphi|$.

Remark. Our translation is motivated by that of iPDL [Degen06].

Translating WK* into PDL

Theorem

$\Vdash_{\mathsf{WK}^*} \varphi$ *iff* $\Vdash_{\mathsf{PDL}} \tau(\varphi)$ *for any* $\varphi \in \mathcal{L}^*$.

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Proof idea

$(\Rightarrow)$: Transform a PDL-model $\mathcal{M}^{\mathsf{PDL}} = \langle W, \rho, \|\cdot\|^{\mathsf{PDL}}\rangle$ into a WK-model

$$\mathcal{M}^{\mathsf{WK}} := \langle W, \rho(i^*), \rho(m), \|\cdot\|^{\mathsf{WK}}\rangle$$

by setting

$$\|p\|^{\mathsf{WK}} := \|[i^*]p\|^{\mathsf{PDL}} \text{ for } p \in \mathrm{Prop} \quad \text{and} \quad \|\bot\|^{\mathsf{WK}} := \emptyset.$$

$\mathcal{M}^{\mathsf{WK}}$ satisfies a formula iff $\mathcal{M}^{\mathsf{PDL}}$ satisfies its translation.

$$
\begin{tikzcd}
\mathsf{PDL} \arrow[r, phantom, "\supseteq"] & \mathsf{K}^* \arrow[r, hook] & \mathsf{CK}^*_{\Box} \arrow[r, phantom, "\subseteq"] & \mathsf{CK}^* \arrow[r, hook, "\checkmark"] & \mathsf{WK}^* \arrow[r, hook, "\checkmark"] & \mathsf{PDL} \\
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\end{tikzcd}
$$

Theorem

Validity of WK^*, CK^* *and* $\mathsf{CK}^*_{\Box}$ *is in* EXPTIME.

Theorem

WK^*, CK^*, *and* $\mathsf{CK}^*_{\Box}$ *have the exponential-size model property.*

Translating K^* into $CK^*_\Box$

1st attempt: double negation translation... does not work

We define the translation by forcing excluded middle:

Definition

The translation $\iota : \mathcal{L}^{K^*} \to \mathcal{L}^*_\Box$ is defined as

$$\iota(\varphi) := \Box^* \bigwedge\{\psi \vee \neg\psi \mid \psi \in \mathit{Subf}(\varphi)\} \to \varphi$$

where $\neg\psi := \psi \to \bot$. $|\iota(\varphi)|$ is polynomial on $|\varphi|$.

Theorem

$\Vdash_{K^*} \varphi$ iff $\Vdash_{CK^*_\Box} \iota(\varphi)$ for any $\varphi \in \mathcal{L}^{K^*}$.

$$\begin{tikzcd}
\mathsf{PDL} \arrow[r, phantom, "\supseteq"] & \mathsf{K}^* \arrow[r, hook, "\checkmark"] & \mathsf{CK}^*_\Box \arrow[r, phantom, "\subseteq"] & \mathsf{CK}^* \arrow[r, hook, "\checkmark"] & \mathsf{WK}^* \arrow[r, hook, "\checkmark"] & \mathsf{PDL} \\
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Theorem

Validity of WK^*, CK^* *and* $\mathsf{CK}^*_\Box$ *is* $\textsc{ExpTime}$*-hard.*

Corollary

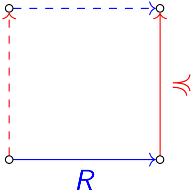
WK^*, CK^* *and* $\mathsf{CK}^*_\Box$ *are* $\textsc{ExpTime}$*-complete.*

Transitive reflexive CS4 and WS4

Definition

A CS4-*model* is a CK-model where R is a preorder and

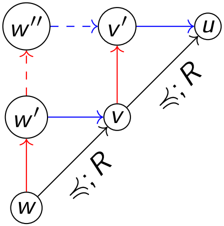
$$\text{if } w \; R \; v \preccurlyeq v', \text{there exists } w' \text{ st } w \preccurlyeq w' \; R \; v'.$$



A WS4-*model* is an infallible CS4-model.

Lemma

If $\mathcal{M}$ is a CS4*-frame, then* $(\preccurlyeq; R)$ *is a preorder.*



Definition

The translation $\kappa : \mathcal{L} \to \mathcal{L}^*$ is defined by letting $\kappa(\varphi)$ be the formula obtained by replacing $\Box$ by $\Box^*$ and $\Diamond$ by $\Diamond^*$.

Embedding CS4 and WS4

Theorem

$\Vdash_{\mathsf{CS4}} \varphi$ *iff* $\Vdash_{\mathsf{CK}^*} \kappa(\varphi)$ *and* $\Vdash_{\mathsf{WS4}} \varphi$ *iff* $\Vdash_{\mathsf{WK}^*} \kappa(\varphi)$ *for* $\varphi \in \mathcal{L}$.

Theorem

$\Vdash_{\mathsf{CS4}} \varphi$ *iff* $\Vdash_{\mathsf{CK}^*} \kappa(\varphi)$ *and* $\Vdash_{\mathsf{WS4}} \varphi$ *iff* $\Vdash_{\mathsf{WK}^*} \kappa(\varphi)$ *for* $\varphi \in \mathcal{L}$.

Proof idea

$(\Rightarrow)$: Transform a CK-model into a CS4-model by duplicating the worlds and extending the modal relation.

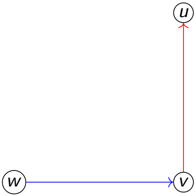


Figure: CK-model

$\Longrightarrow$

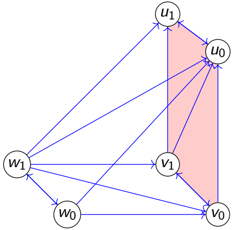


Figure: CS4-model

Idea: 0-worlds relate over R^* while 1-worlds expand accessibility with $(\preccurlyeq;\mathsf{R}^*)^*$-accessible worlds.

Concluding remarks

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1. We defined WK^*, CK^* and $\mathsf{CK}^*_\Box$.
2. Translation methodology into PDL
   $\Rightarrow$ exp-size model property and ExpTime-completeness.

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   $\Rightarrow$ improves NExpTime upper bound [Balbiani21].

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Open questions

1. Deductive calculi for WK^* and CK^*.
2. CS4 is PSpace-complete (lower bound in [Statman79]).

Thanks for your attention!

[Paiva01] G. Bellin, V. D. Paiva, and E. Ritter, *Extended Curry–Howard correspondence for a basic constructive modal logic*, In *Proceedings of the 2nd Workshop on Methods for Modalities (M4M 2001)*, Amsterdam, Netherlands, 2001.

[Wije90] Duminda Wijesekera, *Constructive modal logics I. Annals of Pure and Applied Logic*, 50(3):271–301, 1990.

[Degen06] W. Degen and J. M. Werner, *Towards intuitionistic dynamic logic*, In *Logic and Logical Philosophy*, volume 15, pages 305–324, 2006.

[Celani01] Sergio Celani, *A fragment of intuitionistic dynamic logic. Fundamenta Informaticae*, 46:187–197, 2001.

[Afshari24] Bahareh Afshari, Lide Grotenhuis, Graham E. Leigh, and Lukas Zenger, *Intuitionistic master modality*, In *Advances in Modal Logic*, pages 19–39. College Publications, 2024.

[Statman79] Richard Statman, *Intuitionistic propositional logic is polynomial-space complete. Theoretical Computer Science*, 9(1):67–72, 1979.

[Fischer79] Michael J. Fischer and Richard E. Ladner, *Propositional dynamic logic of regular programs. Journal of Computer and System Sciences*, 18(2):194–211, 1979.

[Pratt80] Vaughan R. Pratt, *A near-optimal method for reasoning about action*, In *Journal of Computer and System Sciences*, 20(2):231–254, 1980.

[Balbiani21] Philippe Balbiani, Martín Diéguez, and David Fernández-Duque, *Some constructive variants of S4 with the finite model property*. In *Proceedings of the 36th Annual ACM/IEEE Symposium on Logic in Computer Science (LICS 2021)*, pages 1–13, IEEE, 2021.

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