An attempt at disentangling logical and semantical necessity

Second Workshop on Worlds and Truth Values, Barcelona

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Semantic justification

- If $\models A$ stands for
  A is true at any possible world.
- and, if the semantics of $\Box A$ is stipulated by
  $\Box A$ is true at some possible world w if and only if A is true at all possible worlds of w.
- Then the Necessitation rule $\frac{A}{\Box A}$ has a clear justification.

A common misconception

- The rule of Necessitation
  If I know that A,
  then,
  I may conclude that $\Box A$*.*
- Wrong application of Necessitation:

$$\dfrac{\dfrac{[\varphi]^1}{\Box\varphi}\,\text{Nec}}{\varphi \to \Box\varphi}\,{\to}\text{I},\ 1 \quad .$$

Epistemic justification of Necessity

- How to interpret modal reasoning $\vdash$
- If $\vdash$ is just an artifact to model $\models$ then as before, Necessitation is clear
- If we try to endow $\vdash$ with an independent epistemic justification for *reasoning* about Necessity, then
- the Rule of Necessity seems to impose some Necessary status of *reasoning/logic*:

  If I can justify the validity of A using my reasoning system
  then
  since this reasoning is necessary
  necessarily A is also justified for my reasoning system

- The conclusion seems to be: logic is necessary
- However, the possible world semantics allows for different possible worlds ruled by different logics

Defining the Language and Derivations

- Language $\mathcal{L}_\Box := p \mid \bot \mid A \wedge A \mid A \vee A \mid A \to A \mid \Box A$
- Set $\text{Form}_\Box$ of formulas in $\mathcal{L}_\Box$
- $(\Gamma, \varphi \subseteq \text{Form}_\Box)$ A *classical derivation* $\mathcal{D}$ from Γ to φ is a sequence of formulas $\varphi_1, \varphi_2, ..., \varphi_k$ s.t $\forall i \in \{1, 2, ..., k\}$:
  - $\varphi_i \in \Gamma$ or
  - φ_i is in the form of a Classical tautology in the language $\mathcal{L}_\Box$ or
  - There is $j, l < i$ such that φ_j is of the form $\varphi_l \to \varphi_i$
  - $\varphi_k = \varphi$.

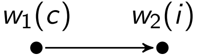
Defining the Language and Derivations

- ($\Gamma, \varphi \subseteq \text{Form}_{\Box}$) An *Intuitionistic derivation* $\mathcal{D}$ from Γ to φ is a sequence of formulas $\varphi_1, \varphi_2, ..., \varphi_k$ s.t $\forall i \in \{1, 2, ..., k\}$:
  - $\varphi_i \in \Gamma$ or
  - φ_i is in the form of an Intuitionistic tautology in the language $\mathcal{L}_{\Box}$ or
  - There is $j, l < i$ such that φ_j is of the form $\varphi_l \to \varphi_i$
  - $\varphi_k = \varphi$.
- $\vdash_{\mathrm{c}}^{\mathcal{L}_{\Box}}$ / $\vdash_{\mathrm{i}}^{\mathcal{L}_{\Box}}$ represents a classical/intuitionistic derivation in $\mathcal{L}_{\Box}$
- $\overline{\mathrm{T}}^{c_{\Box}}/\overline{\mathrm{T}}^{i_{\Box}}$ is the closure of T over $\vdash_{\mathrm{c}}^{\mathcal{L}_{\Box}}$ / $\vdash_{\mathrm{i}}^{\mathcal{L}_{\Box}}$

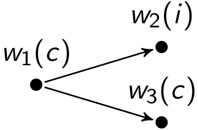
Defining the models

- A Mixed model is a tuple $\mathcal{M} := \langle W, R, e\rangle$ where $\langle W, R\rangle$ is a Kripke Frame and e is an *extension* $e : W \to \mathcal{P}(\text{Form}_\Box) \times \{i, c\}$ $\big(\text{denoted } e(w) = \langle T_w, l_w\rangle\big)$ such that:
  1. $\bot \notin T_w$;
  2. $T_w \vdash^{\mathcal{L}\Box}_{l_w} \varphi \Rightarrow \varphi \in T_w$;
  3. $\Box\varphi \in T_w \iff \forall v(wRv \Rightarrow \varphi \in T_w)$;
  4. $\neg\Box\varphi \in T_w \iff \exists u(wRu \wedge \varphi \notin T_u)$.

First examples of Mixed Models

- 

  - $F_{w_2} = \overline{\{p, q\} \cup \{\Box\varphi \mid \varphi \in \mathbf{Form}_\Box\}}^{i\Box}$;
  - $F_{w_1} = \overline{\{\neg q\} \cup \{\Box\varphi \mid \varphi \in F_{w_2}\} \cup \{\neg\Box\psi \mid \psi \in \mathbf{Form}_\Box / F_{w_2}\}}^{c\Box}$



-
  - $F_{w_3} = \overline{\{p\} \cup \{\Box\varphi \mid \varphi \in \mathbf{Form}_\Box\}}^{c\Box}$
  - $F_{w_2} = \overline{\{p, q\} \cup \{\Box\varphi \mid \varphi \in \mathbf{Form}_\Box\}}^{i\Box}$
  - $F_{w_1} =$ $\overline{\{\neg p \vee q\} \cup \{\Box\varphi \mid \varphi \in F_{w_2} \cap F_{w_3}\} \cup \{\neg\Box\psi \mid \psi \in \mathbf{Form}_\Box / F_{w_2} \cap F_{w_3}\}}^{c\Box}$

Intuitionistic logic and Modal logic

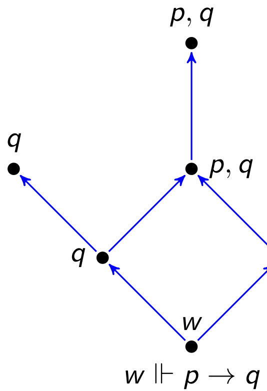
- Intuitionistic propositional logic IPC:
  - Language: $A ::= p \mid \bot \mid A \wedge A \mid A \vee A \mid A \rightarrow A$
  - Intuitionistic tautologies
  - Rules: Modus Ponens

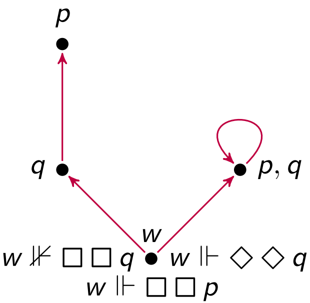
- Classical modal logic K:
  - Language: $A ::= p \mid \bot \mid A \wedge A \mid A \vee A \mid A \rightarrow A \mid \Box A \mid \Diamond A$
  - Classical tautologies
  - K-axiom: $\Box(A \rightarrow B) \rightarrow \Box A \rightarrow \Box B$
  - Rules: Modus Ponens and Necessitation

Intuitionistic logic and Modal logic: Semantics

- Kripke semantics for IPC:
  - $M = (W, \leq, V)$ (Monotonicity w.r.t. V)
  - $M, w \Vdash A \to B$ iff for all $v \geq w$: $M, v \Vdash A$ implies $M, v \Vdash B$
- Possible world semantics for K:
  - $M = (W, R, V)$
  - $M, w \Vdash \Box A$ iff for all v s.t. wRv: $M, v \Vdash A$
    $M, w \Vdash \Diamond A$ iff there exists v s.t. wRv and $M, v \Vdash A$

Some examples





Intuitionistic modal logics

Quest to intuitionistic meaning of $\Box$ and $\Diamond$

Classical consequences of the K-axiom:

(k1) $\Box(A \to B) \to \Box A \to \Box B$
(k2) $\Box(A \to B) \to \Diamond A \to \Diamond B$
(k3) $\Diamond(A \vee B) \to \Diamond A \vee \Diamond B$
(k4) $(\Diamond A \to \Box B) \to \Box(A \to B)$
(k5) $\neg\Diamond\bot$

Different intuitionistic/constructive modal logics:

- iK := IPC + (k1)
- CK := IPC + (k1) + (k2)
- IK := IPC + (k1) + (k2) + (k3) + (k4) + (k5)
- ...

Intermezzo

Theorem

iK *and* CK *prove the same* $\Diamond$*-free theorems*

Theorem (Das&Marin, 2023)

iK *and* IK *do not have the same* $\Diamond$*-free theorems*

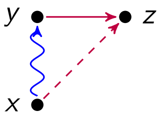
For example:

$$\neg\neg\Box\bot \to \Box\bot \in \mathsf{IK} \setminus \mathsf{iK}$$

$$\neg\neg\Box p \to \Box p \in \mathsf{IK} \setminus \mathsf{iK}$$

Birelational semantics for iK

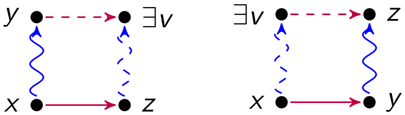
- $M = (W, \leq, R, V)$ (Monotonicity w.r.t. V)
- Frame property (F0):



- $M, w \Vdash \Box A$ iff for all v s.t. wRv: $M, v \Vdash A$

Birelational semantics for IK

- $M = (W, \leq, R, V)$ (Monotonicity w.r.t. V)
- Frame properties (F1) and (F2):



- $M, w \Vdash \Box A$ iff for all $w' \geq w$ and all v s.t. $w' R v$: $M, v \Vdash A$
  $M, w \Vdash \Diamond A$ iff there exists v s.t. $w R v$ and $M, v \Vdash A$

Concrete models

- Concrete Models:
  From a KF $F = \langle W, R \rangle$ and function $\lambda : W \to \{c, i\}$, we assign to each $w \in W$ a rooted intuitionistic Kripke Model $\langle U_w, \leq_w, V_w \rangle$(root: $\overline{w} \in U_w$) st $\lambda(w) = c \Rightarrow U_w = \{\overline{w}\}$
- $\Vdash$ is defined on $\Theta := \bigcup_{w \in W} U_w$ (for $x \in U_w$):
  1. $x \nVdash \bot$ and $x \Vdash \top$;
  2. $x \Vdash p$ iff $x \in V_w(p)$;
  3. $x \Vdash A \wedge B$ iff $x \Vdash A$ and $x \Vdash B$;
  4. $x \Vdash A \vee B$ iff $x \Vdash A$ or $x \Vdash B$;
  5. $x \Vdash A \to B$ iff $\forall y \in U_w (x \leq y \to y \nVdash A$ or $y \Vdash B)$;
  6. $x \Vdash \neg A$ iff $x \Vdash A \to \bot$;
  7. $x \Vdash \Box A$ iff $\forall v (wRv \to \overline{v} \Vdash A)$.

Predicate models for IK

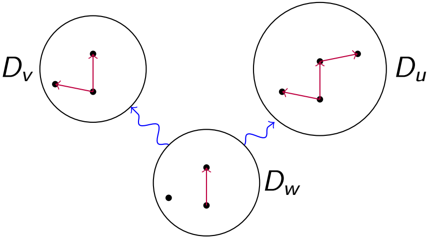
- iK embeds into K via the Kuroda translation,
- IK embeds into K via the Gödel-Gentzen translation, moreover,
- IK embeds into IQC by the *standard translation*:

$$ST(A) := \forall x ST_x(A) \text{ with } \quad \begin{array}{l} ST_x(\Box A) := \forall y(xRy \to ST_y(A)) \\ ST_x(\Diamond A) := \exists y(xRy \wedge ST_y(A)) \end{array}$$

- Predicate models $\Rightarrow$ birelational semantics with (F1) and (F2)

Predicate models for IK

We observe that Concrete Mix Models are dual to predicate models of IK!



$M, w \Vdash \forall x\varphi$ iff for all $w' \geq w$ and all $d \in D_{w'}$: $M, w' \Vdash \varphi[x/d]$
$M, w \Vdash \exists x\varphi$ iff there exists $d \in D_w$ s.t. $M, w \Vdash \varphi[x/d]$

Conjecture for Concrete models

- Theorem: Let $\Gamma_w := \{\varphi \mid \text{w} \Vdash \varphi\}$. The KF F together with the extention e defined $e(w) = \langle \Gamma_w ; \lambda(w) \rangle$ defines a Mixed Model, called Concrete Model.
- Example of a non-concrete Mixed Model: $F = \langle \{w\}, R \rangle$, $R = \varnothing$, $l_w = c$, $T_w = \overline{\{p \vee q\} \cup \{\Box\varphi \mid \varphi \in \textbf{Form}_\Box\}}^c$
- Conjecture: The class $\mathcal{CM}$ of all Concrete Models is the class of all Mixed Models such that for all $M \in \mathcal{CM}$, $w \in M$:
  - If $l_w = c$, T_w is a maximal theory
  - If $l_w = i$, T_w is a prime theory ($\varphi \vee \psi \in T_w \Rightarrow \varphi \in T_w$ or $\psi \in T_w$).

Soundness for $\mathcal{MM}$

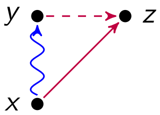
- Soundness: iK + $\Box A \vee \neg\Box A$ is sound with respect to the class $\mathcal{MM}$ of all Mixed Models.
- Results of interest:
  - (Necessitation)$M \vDash A$ implies $M \vDash \Box A$;
  - (Distributivity)$M \vDash \Box(A \to B) \to (\Box A \to \Box B)$.

Quick proof of Distributivity(k-axiom)

- $(M \in \mathcal{MM})$ We want $M \vDash \Box(A \to B) \to (\Box A \to \Box B)$ (i.e $\forall w \in M, \Box(A \to B) \to (\Box A \to \Box B) \in T_w$)
  - $(\mathbb{A}\ \Box(A \to B) \in F_w)$
    - If $\Box A \in F_w$, $\forall y \in M(A, A \to B \in F_y \Rightarrow B \in F_y) \Rightarrow \Box B \in F_w \Rightarrow \Box(A \to B) \to (\Box A \to \Box B)$
    - If $\Box A \notin F_w$, $\Box A \to \bot \in F_w$, and by reductio ad absurdum, $\Box A \to \Box B \in F_w \Rightarrow \Box(A \to B) \to (\Box A \to \Box B) \in T_w$
  - $((\mathbb{A}\ \Box(A \to B) \notin F_w)$, then $\Box(A \to B) \to \bot \in F_w$ and by reductio ad absurdum, $\Box(A \to B) \to (\Box A \to \Box B) \in T_w$

Frame condition and possible completeness

- Frame condition for $\Box A \vee \neg\Box A$ (F3):



- Completeness of $\mathcal{MM}$ with regards to iK+$\Box A \vee \neg\Box A$ Would require:
  - Completeness of Birelational models $\mathcal{BM}$ with (F0+F3) with regards to iK+$\Box A \vee \neg\Box A$
  - Transition from $\mathcal{BM}$ to $\mathcal{MM}$ Models (Unraveling)

Combining various logics

- Incomparable, for example
  - Gödel-Dummett logic LC of linear Kripke frames

$$(p \to q) \vee (q \to p)$$

  - Intuitionistic Logic of bounded depth two BD_2

$$p \vee (p \to (q \vee \neg q))$$

- Many valued
- Etc.

On the structure of time

- Locally, time can behave differently than globally
- Universal time versus black-hole horizon, etc.
- combining different temporal logics

Thank you for your attention and feedback