Interpretability Logics and Ultrafilter Extensions

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Lectures on Logic and its Mathematical Aspects
LLAMA Seminar
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Overview

- Interpretability Logics and Ultrafilter Extensions

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- Revisit Ultrafilter Extensions for modal logics

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  - $\mathcal{M}, x \Vdash \Diamond\varphi \iff \exists y{\in}W \left(xRy \wedge \mathcal{M}, y \Vdash \varphi\right)$

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  - Binding preference from strong to weaker: $\{\neg, \Diamond\}, \{\wedge, \vee\}, \to$

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- Without much further motivation:
- Modally saturated models are very important!
  van Benthem modal definability characterisation, Goldblatt-Thomason frame definability characterisation, ...

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- Spoiler: this theorem also holds for interpretability logics
- where things are a bit more subtle

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- Let us revisit how they work

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- Idea 1 + Idea 2 $\implies$ worlds are ultrafilters over W

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- We thus define: $fR^{\mathrm{ue}}g \iff \forall A \in g \ R^{-1}(A) \iff R^{-1}g \subseteq f$

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  - $f\, R^{\mathrm{ue}}\, g \quad :\Longleftrightarrow \quad R^{-1}g \subseteq f.$
- Given $x \in W$ we denote the *principal ultrafilter generated by* x by Π_x
- Thus, $\Pi_x := \{A \subseteq W \mid x \in A\}$
- $\pi : x \mapsto \Pi_x$ defines an injection $\pi : W \to W^{\mathrm{ue}}$
- Not hard to see: $xRy \quad \Longleftrightarrow \Pi_x R^{\mathrm{ue}} \Pi_y$
- Our running example...

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- Not hard to see: $\text{ue}(\mathcal{M})$ is modally saturated

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- Who would call herself a specialist on possibility logic...

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- Redundancy:

$$\models \Diamond\varphi \quad \leftrightarrow \quad \neg(\varphi \rhd \bot)$$

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- Especially *modally saturated models*

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- and see that we essentially need copies of worlds/maximal-consistent-sets/ultrafilters

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- We need to mark the role of ultrafilters:
  - Classical theory, de Jongh-Veltman works with criticality labels (one promise at the time)
  - Modern approach: assuring labels (possibly infinitely many promises simultaneously)

Algebraic treatment of $\rhd$

- Direct translation:
$[\![(\varphi \rhd \psi)]\!]_{\mathcal{M}} = \{x \mid \forall y \Big(xRy \wedge \mathcal{M}, y \Vdash \varphi \to \exists z \, (yS_x z \wedge \mathcal{M}, z \Vdash \psi)\Big)\}$

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- This should be a main ingredient to define $gS_f^{\mathrm{ue}}h$ in the ultrafilter extension

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- Algebraic counterpart: in the definition of $f \prec_S g$ we can take S to be a proper filter!

Ultrafilter extensions

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- Running example

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Details can be found at

F.F. González, J.J. Joosten, V.N. Arroyo, C.P. Brogi, *Ultrafilter Extensions for Veltman Semantics*, **arXiv:2603.16754 [math.LO]**, (2026).

Thank you