Formalised provability in constructive arithmetic

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Formalised provability and applications

- Provability is a central notion in logic and metamathematics
- For theories like PA we can write a Σ_1 predicate $\Box_{\mathsf{PA}}(\cdot)$ such that

$$\mathsf{PA} \vdash \varphi \quad \Longleftrightarrow \quad \mathbb{N} \models \Box_{\mathsf{PA}}(\ulcorner \varphi \urcorner)$$

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Theorem

The $\Box_{\mathsf{PA}}(\cdot)$ predicate is Σ^0_1-complete. That is, for each c.e. set A, there is an arithmetical formula $\rho_A(x)$ such that

$$A = \{n \in \mathbb{N} \mid \mathbb{N} \models \Box_{\mathsf{PA}}\Big(\rho_A(n)\Big)\}.$$

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- Löb's Theorem:

  If $\mathsf{PA} \vdash \Box_{\mathsf{PA}}(\ulcorner A \urcorner) \to A$, *then* $\mathsf{PA} \vdash A$, *for any* PA *formula* A

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- Formalised Löb's Theorem (ignoring GNs):

$$\mathsf{PA} \vdash \Box_{\mathsf{PA}}\Big(\Box_{\mathsf{PA}} A \to A\Big) \to \Box_{\mathsf{PA}} A$$

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- Characterise all provably structural properties in two steps
  - $\mathcal{L}_\Box$ with $\mathrm{Form}_\Box := \bot \mid \mathrm{Prop} \mid \mathrm{Form}_\Box \to \mathrm{Form}_\Box \mid \Box\mathrm{Form}_\Box$
  - Define a denotation of $\mathcal{L}_\Box$ formulas inside the $\mathcal{L}_{\mathsf{PA}}$ formulas

Arithmetical realizations

An arithmetical realization is any function $(\cdot)^\star$ taking:

$$\text{formulas in } \mathcal{L}_\Box \to \text{sentences in } \mathcal{L}_{\mathsf{PA}}$$
$$\text{propositional variables} \to \text{arithmetical sentences}$$
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Clearly, for any realization $(\cdot)^\star$ we have for example

$$\mathsf{PA} \vdash \Big(\Box(p \to q) \to (\Box p \to \Box q)\Big)^\star$$

since

$$\mathsf{PA} \vdash \Box_{\mathsf{PA}}(p^\star \to q^\star) \to (\Box_{\mathsf{PA}} p^\star \to \Box_{\mathsf{PA}} q^\star)$$

regardless of $(\cdot)^\star$

The Provability Logic of a Theory

- For a c.e. theory T we define

$$\mathsf{PL}(T) := \{\varphi \in \mathcal{L}_\Box \mid \text{for any } (\cdot)^\star, \text{ we have } T \vdash (\varphi)^\star\}$$

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A candidate

- GL is the normal modal logic with axioms
  - All classical logical tautologies in $\mathcal{L}_\Box$ like $\Box p \vee \neg\Box p$, etc.
  - All distributions axioms: $\Box(A \to B) \to (\Box A \to \Box B)$,
  - All Löb axioms: $\Box(\Box A \to A) \to \Box A$.
- and rules
  - Modus Ponens $\dfrac{A \to B \quad A}{B}$,
  - Necessitation $\dfrac{A}{\Box A}$.

Solovay's Theorem

Theorem (Solovay, 1976)

Let $\varphi \in \mathcal{L}_\Box$*. Then:*

$$\mathsf{GL} \vdash \varphi$$

$$\Updownarrow$$

$$\mathsf{PA} \vdash (\varphi)^\star \textit{ for any arithmetical realization } (\cdot)^\star$$

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Thus, even though PL(PA) is *prima facie* of complexity Π^0_2, it allows for a decidable description

$$\mathsf{GL} = \{\varphi \in \mathcal{L}_\Box \mid \text{for any } (\cdot)^\star, \text{ we have } \mathsf{PA} \vdash (\varphi)^\star\}$$

of complexity PSPACE.

True provability logic

- $\mathsf{PA} \nvdash \Box_{\mathsf{PA}}(\ulcorner 0 = 1 \urcorner) \to 0 = 1$
- $\mathbb{N} \models \Box_{\mathsf{PA}}(\ulcorner \varphi \urcorner) \to \varphi$ for whatever sentence φ

For a c.e. theory T we define

$$\mathsf{TPL}(T) := \{\varphi \in \mathcal{L}_{\Box} \mid \text{for any } (\cdot)^{\star}, \text{ we have } \mathbb{N} \models (\varphi)^{\star}\}$$

A priori, complexity above true arithmetic.
However,

$$\mathsf{TPL}(\mathsf{PA}) = \mathsf{GLS}.$$

Here GLS is axiomatised by all theorems of GL and all reflection axioms $\Box A \to A$ with MP as the only rule.

Solovay for quantified modal logic?

Let $\mathcal{L}_{\Box,\forall}$ be the language of relational quantified modal logic:

$$\top\text{, relation symbols, boolean connectives, } \forall x\text{, and } \Box$$

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Define arithmetical realizations $(\cdot)^\bullet$ for $\mathcal{L}_{\Box,\forall}$:

$$\text{formulas in } \mathcal{L}_{\Box,\forall} \to \text{formulas in } \mathcal{L}_{\mathsf{PA}}$$

$$n\text{-ary relation symbols} \to \text{arithmetical formulas with } n \text{ free variables}$$

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$$\mathrm{QPL}(T) := \{\varphi \in \mathcal{L}_{\Box,\forall} \mid \text{for any } (\cdot)^\bullet,\ \text{we have } T \vdash (\varphi)^\bullet\}$$

and

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Example: $\Box\forall x P(x) \to \forall x \Box P(\dot{x})$

Degenerate Quantified Provability Logics

If we define $\mathsf{QL}(T) = \{\varphi \in \mathcal{L}_{\mathsf{pred}} \mid \text{for any } (\cdot)^\bullet, \text{ we have } T \vdash (\varphi)^\bullet\}$, then it is not hard to see that $\mathsf{CQC} = \mathsf{QL}(\mathsf{PA})$.

Proof:

- $\subseteq$ if $\pi \vdash_{\mathsf{CQC}} \varphi$, then also $\pi^\bullet \vdash_{\mathsf{CQC}} \varphi^\bullet$, whence $\pi^\bullet \vdash_{\mathsf{PA}} \varphi^\bullet$
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$$\mathsf{QPL}(\mathsf{PA} + \mathsf{Incon}(\mathsf{PA})) = \mathsf{CQC} + \Box\bot$$

Negative results

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Theorem (Vardanyan, 1986 and McGee, 1985)

$$\{\textit{closed } \varphi \in \mathcal{L}_{\Box,\forall} \mid \textit{for any } (\cdot)^\bullet,\ \textit{we have } \mathsf{PA} \vdash (\varphi)^\bullet\}$$

is Π^0_2*-complete.*

Theorem (Artemov, 1985)

TQPL(PA) *is not arithmetical.*

Theorem (Vardanyan, 1985)

TQPL(PA) *is* Π^0_1 *complete in true arithmetic.*

Artemov's Lemma

- Let $F \in \mathcal{L}_{\text{PA}}$ be a formula

$\mathcal{L}_{PA}$

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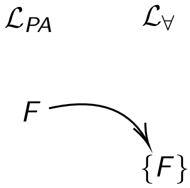
- Let $F \in \mathcal{L}_{\mathsf{PA}}$ be a formula
- Replace arithmetical symbols $0, +1, +, \times, =$ with predicates Z, S, A, M, E, obtaining $\{F\} \in \mathcal{L}_{\forall}$



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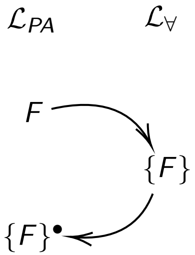

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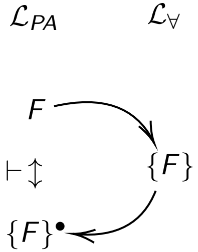
When are F and $\{F\}^{\bullet}$ equivalent over PA?



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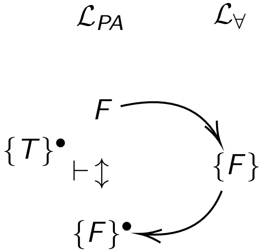

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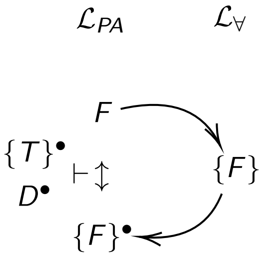

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- ... and under $D^{\bullet}$ to get recursive $A^{\bullet}$ and $M^{\bullet}$

$$
\begin{aligned}
D := \Diamond\top \wedge \\
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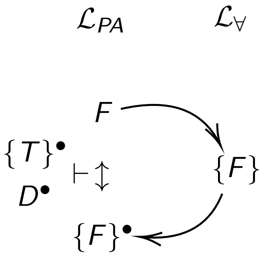

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- ... and under $D^\bullet$ to get recursive $A^\bullet$ and $M^\bullet$
- By Tennenbaum's Theorem the model induced by $(\cdot)^\bullet$ is standard, hence $\mathbb{N} \models S \iff (\{T\} \wedge D \to \{S\}) \in \mathsf{TQPL(PA)}$

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One easily sees that $\mathsf{QPL}(\mathsf{PA} + \Box_{\mathsf{PA}}\bot)$ is r.e., but it seems that $\mathsf{QPL}(\mathsf{PA} + \Box_{\mathsf{PA}}\Box_{\mathsf{PA}}\bot)$ is also Π^0_2-complete.

Theorem (Visser, de Jonge, 2006)

$\mathsf{QPL}(T)$ *is* Π^0_2 *complete for any* Σ_1*-sound theory* T *extending* EA.

Archive for Mathematical Logic 2006: No Escape from Vardanyan's

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- Conjectured to be PSPACE

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    - $$\mathsf{HA} \vdash \varphi \iff \mathsf{HA} \vdash \Box_{\mathsf{HA}}\varphi$$

Propositional logic of HA

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- Pretty stable but

$$\mathrm{PropL}(\mathrm{HA} + \mathrm{CT} + \mathrm{MP})$$

  is unknown.

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- The formalised disjunction property is equivalent over HA to RFN(HA)

Markov's principle

- Markov's Rule is admissible for HA

$$\mathsf{HA} \vdash \neg\neg\pi \quad \Rightarrow \quad \mathsf{HA} \vdash \pi$$

For $\pi \in \Pi^0_2$

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- And more in general

$$\Box\Big(\neg\neg(\Box A \to \bigvee_i \Box A_i)\Big) \to \Box(\Box A \to \bigvee_i \Box A_i)$$

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Admissible rules

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- For IPC the situation is very different where an example of non-trivial admissible rule is the so-called *Independence of premise* principle

$$\frac{\neg A \to B \vee C}{(\neg A \to B) \vee (\neg A \to C)}$$

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- Iemhoff: characterisation in terms of Visser rules
- If $\frac{A}{B}$ is admissible for IPC, then $\Box A \to \Box B \in \mathsf{PL}(\mathsf{HA})$

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- Visser's rule is admissible for IPC and in Iemhoff's sense these rules generate all admissible rules.

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- It seems that Vardanyan can be extended to QPL(HA).

Restricted signatures and logics: RC_1

Restrict $\mathcal{L}_\Box$ to the strictly positive fragment $\mathcal{L}_\Diamond$:

$$\mathcal{L}_\Diamond ::= \top \mid \varphi \wedge \varphi \mid \Diamond\varphi$$

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Define a calculus RC_1 with statements $\varphi \vdash_{\mathsf{RC}_1} \psi$ where:

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RC_1: Axioms and rules

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RC_1 Main result

Theorem (Dashkov, Beklemishev)

Let $\varphi, \psi \in \mathcal{L}_\Diamond$. *Then:*

$$\mathsf{GL} \vdash \varphi \to \psi$$

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$$\mathsf{PA} \vdash (\varphi \to \psi)^\star \text{ for any arithmetical realization } (\cdot)^\star$$

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Even though the fragment looks poor, its polymodal (up to ω) version suffices for an ordinal notation up to ε_0 and it can perform the main computations of an ordinal analyses of PA and subsystems

Restricted signatures and logics: QRC_1

Restrict $\mathcal{L}_{\Box,\forall}$ to the strictly positive fragment $\mathcal{L}_{\Diamond,\forall}$:

$$\text{Terms} ::= \text{Variables} \mid \text{Constants}$$

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QRC_1 Main result

Theorem (de Almeida Borges, JjJ)

Let $\varphi, \psi \in \mathcal{L}_{\Diamond,\forall}$. *Then:*

$$\varphi \vdash_{\mathsf{QRC}_1} \psi$$

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$\mathsf{PA} \vdash (\varphi \to \psi)^\bullet$ *for any arithmetical realization* $(\cdot)^\bullet$

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Theorem (Positive fragment)

Let φ *and* ψ *be* QRC_1 *formulas (no constants) and let* QS *be any logic between* QK4 *and* QGL. *Then* $\varphi \vdash_{\mathsf{QRC}_1} \psi$ *if and only if* $\mathsf{QS} \vdash \varphi \to \psi$.

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   - TQPL(PA) is Π^0_1-complete in $(0)^\omega$ (non-arithmetical)
   - Advanced conjecture:: TQPL(PA)+ is decidable:
     $(A \to B) \in \mathrm{TQPL(PA)} \quad \Leftrightarrow A \wedge Q^n(A) \vdash_{\mathrm{QRC}_1} B$ for n large enough
     where Q^n denotes n times iterated consistency

Older escapes to Vardanyan

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Older escapes to Vardanyan

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- Yavorski, add $\Box A \to \Box \forall x A$

Some provable and unprovable statements

$$\Diamond \forall x \, \varphi \vdash \forall x \, \Diamond \varphi$$

$$\forall x \, \Diamond \varphi \nvdash \Diamond \forall x \, \varphi$$

$$\frac{\varphi \vdash \psi[x \leftarrow c]}{\varphi \vdash \forall x \, \psi}$$

x not free in φ and c not in φ nor ψ

Recall that RC_ω allows for ordinal notations up to ε_0 and that it caters Π^0_1 ordinal analyses.

Can be extended to RC_Λ.

Relational models

Kripke models where:

- each world w is a first-order model with a finite domain D
- the domain D is the same for every world
- each constant symbol c and relational symbol S has a denotation at each world
- there is a transitive relation R between worlds
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- we use assignments g : Variables $\to D$ to interpret variables
- we abuse notation and define $g(c) :=$ denotation(c) for all assignments g and constants c

Satisfaction

Let g be a w-assignment.

$$\mathcal{M}, w \Vdash^g S(t,u) \iff \langle g(t), g(u) \rangle \in \mathsf{denotation}_w(S)$$

$\mathcal{M}, w \Vdash^g \Diamond\varphi \iff$

there is a world v such that wRv and $\mathcal{M}, v \Vdash^g \varphi$

$\mathcal{M}, w \Vdash^g \forall x\, \varphi \iff$

for all assignments $h \sim_x g$, we have $\mathcal{M}, w \Vdash^h \varphi$

Relational soundness

Theorem (Relational soundness)

If $\varphi \vdash \psi$, then for any model $\mathcal{M}$, world w, and assignment g:

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Countermodels with arbitrarily large domains are needed.

$$\forall x, y\, S(x, x, y) \wedge \forall x, y\, S(x, y, x) \wedge \forall x, y\, S(y, x, x) \vdash \forall x, y, z\, S(x, y, z)$$

is unprovable in QRC_1, but satisfied by every world with at most two domain elements.

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is unprovable in QRC_1, but satisfied by every world with at most two domain elements.

Can be extended to arbitrary n.

Relational completeness

Theorem (Relational completeness)

If $\varphi \nvdash \psi$, then there is a finite model $\mathcal{M}$, a world w, and an assignment g such that:

$$\mathcal{M}, w \Vdash^g \varphi \quad \textit{and} \quad \mathcal{M}, w \nVdash^g \psi.$$

Since QRC_1 has the finite model property (finite number of worlds with finite constant domain), it is decidable.

Arithmetical completeness proof

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$$\mathrm{QRC}_1 \supseteq \{\varphi \vdash \psi \mid \textit{for any } (\cdot)^*,\ \textit{we have } T \vdash (\varphi \vdash \psi)^*\}$$

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- Conclude (using external reflection) that

$$T \vdash \chi^\bullet[y \leftarrow \ulcorner g(x) \urcorner] \quad \Leftrightarrow \quad 1 \Vdash \chi^\bullet[y \leftarrow \ulcorner g(x) \urcorner]$$

for relevant χ whence $\mathrm{PA} \nvdash (\varphi \to \psi)^\bullet[y \leftarrow \ulcorner g(x) \urcorner]$

Main results

Theorem (AdAB, DdJ, JjJ, AV)

Let $\varphi, \psi \in \mathcal{L}_{\Diamond,\forall}$. *Then:*

$$\varphi \vdash_{\mathsf{QRC_1}} \psi$$
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- Recall that PL(HA) was a long-standing open problem

Semi-closure

- HA $\nvdash \neg\forall x\,\varphi \leftrightarrow \exists x\,\neg\varphi$.

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- But: $\mathsf{HA} \vdash \Box_{\mathsf{HA}} S \leftrightarrow \Box_{\mathsf{HA}} \neg\neg S$ for $S \in \Sigma_1$
- Trick: employ Π_2-conservativity between HA and PA where we have $\mathsf{HA} \vdash \forall A\ (\Box_{\mathsf{HA}} A \to \Box_{\mathsf{PA}} A)$ *for any* A.

Semi-closure

Lemma

- $\mathsf{HA} \vdash \forall S \in \Sigma_1 \ \Box_{\mathsf{HA}} S \leftrightarrow \Box_{\mathsf{HA}} \neg\neg S$
- $\mathsf{HA} \vdash \forall S \in \Sigma_1 \ (\Box_{\mathsf{HA}} \forall x \neg\neg S \leftrightarrow \Box_{\mathsf{HA}} \forall x\, S).$

The negation of a Π_1 sentence is equivalent to the double negation of a Σ_1 sentence over HA:

Lemma

$$\begin{aligned} \mathsf{HA} \vdash \neg \forall x\, D &\leftrightarrow \neg \forall x \neg\neg D \\ &\leftrightarrow \neg\neg \exists x \neg D \end{aligned} \tag{1}$$

where clearly $\exists x \neg D \in \Sigma_1$.

Lemma

$\mathsf{HA} \vdash \forall A \in \Sigma_2 \ (\Diamond_{\mathsf{HA}} A \leftrightarrow \Diamond_{\mathsf{PA}} A).$

Proof.

In HA, fixing $A \in \Sigma_2$ with $A = \exists x\, P.$ and $S \in \Sigma_1$ so that

$$\mathsf{HA} \vdash \neg P \leftrightarrow \neg\neg S. \tag{2}$$

$$\begin{aligned}
\Diamond_{\mathsf{HA}} A &\leftrightarrow \neg\Box_{\mathsf{HA}}\neg A \\
&\leftrightarrow \neg\Box_{\mathsf{HA}}\neg\exists x\, P \\
&\leftrightarrow \neg\Box_{\mathsf{HA}}\forall x\, \neg P \\
&\leftrightarrow \neg\Box_{\mathsf{HA}}\forall x\, \neg\neg S && \text{by (2)} \\
&\leftrightarrow \neg\Box_{\mathsf{HA}}\forall x\, S \\
&\leftrightarrow \neg\Box_{\mathsf{PA}}\forall x\, S \\
&\leftrightarrow \neg\Box_{\mathsf{PA}}\neg\neg\forall x\, S \\
&\leftrightarrow \Diamond_{\mathsf{PA}}\neg\forall x\, S \\
&\leftrightarrow \Diamond_{\mathsf{PA}}\exists x\, \neg S \\
&\leftrightarrow \Diamond_{\mathsf{PA}} A && \text{by (2).}
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Theorem

$A \vdash_{\mathrm{RC}_1} B$ if and only if for all realizations $\cdot^*$ we have $\mathrm{HA} \vdash (A \to B)^*$.

Proof.

(Completeness) Assume $A \nvdash_{\mathrm{RC}_1} B$. Embed the extended counter model into arithmetic using the PA Solovay function, which will be our arithmetical interpretation, $\cdot^{\circledast}$.

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Whence $\mathrm{PA} \vdash A^{\circledast_{\mathrm{PA}}} \to B^{\circledast_{\mathrm{PA}}}$,

This contradicts completeness of RC_1 w.r.t. PA. □

In summary

- PL(HA) finally settled but lacks an easy axiomatisation
- Strictly positive fragment has an easy axiomatisation with RC
- There is no quantified provability logic with $\mathcal{L}_{\Box,\forall}$

QRC_1:

- quantified, strictly positive provability logic with $\mathcal{L}_{\Diamond,\forall}$
- decidable
- sound and complete w.r.t. relational semantics (with constant domain models!)
- sound and complete w.r.t. arithmetical semantics
- for all sound r.e. theories extending $\mathsf{I}\Sigma_1$
- Both for HA and PA

Forthcoming research

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- Applications to Π^0_1 ordinal analysis?

Thank you

Further Reading I

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