Ordinal analysis based on iterated reflection

First order and beyond

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- $$\begin{array}{lcll} T^0 & := & T; & \\ T^{\alpha+1} & := & T^\alpha + \mathrm{Con}(T^\alpha); & \\ T^\lambda & := & \bigcup_{\alpha<\lambda} T^\alpha & \text{for limit } \lambda < \Gamma. \end{array}$$

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  - $T_n^\lambda := \bigcup_{\alpha<\lambda} T_n^\alpha$ for limit λ.

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- $\omega{-}\mathsf{Con}^G(\mathrm{PA}) \equiv T_2^{\varepsilon_1}$

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- **Proposition:** $T + \langle 1 \rangle_T \top$ is a Π_1 conservative extension of $T + \{\langle 0 \rangle^k_T \top \mid k \in \omega\}$.

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[\xi](A \to B) \to ([\xi]A \to [\xi]B) & \\
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The rules of inference are Modus Ponens and necessitation for each modality: $\frac{\psi}{[\zeta]\psi}$.

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- **Theorem**
  $\mathrm{EA} + \langle n+1 \rangle_{\mathrm{EA}} \top \;\equiv\; \mathrm{EA} + \mathtt{RFN}_{\Sigma_{n+1}}(\mathrm{EA}) \;\equiv\; \mathrm{I}\Sigma_n.$

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- **Definition** By $o_\alpha(A)$ we denote the order type of A under $<_\alpha$ and we write $o(A)$ instead of $o_0(A)$.

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- Worms of GLP_ω are known to be useful for Turing progressions:
- **Proposition** (Beklemishev) For each ordinal $\alpha < \varepsilon_0$ there is some GLP_ω-worm A such that $o(A) = \alpha$, and $T + A$ is Π_1 equivalent to T_α.

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  - $o(000) = o(0) + 1 + o(0) = 1 + 1 + 1 = 3$;
  - $o(0000) = o(0) + 1 + o(00) = 1 + 1 + 2 = 4$, etc.;
  - $o(1) = o(1 \uparrow 0) = e^1(o(0)) = e^1(1) = \omega^1 = \omega$;
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- Ignatiev's model can be interpreted as representing 'natural' theories!

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- This yields a roadmap to conservation results!

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- Ignatiev's model $\mathcal{I}$ generalized (DFD, JjJ, 2013 JSL)

Beklemishev's autonomous worm notation

1	()	2	()()
ω	(())	$\omega + \omega$	(())()(())
ε_0	((()))	ω^{ε_0+1}	(())((()))

Fernandez-Duque's Spiders

ω	$\left({}^{()} \right)$	$\varphi_{\omega_1^{CK}}(1)$	$\left(\begin{smallmatrix} () \\ () \end{smallmatrix} \right)$
ω_1^{CK}	$\left({}_{()} \right)$	$\omega_3^{CK} + \omega_1^{CK}$	$\left({}_{()} \right) () \left({}_{()()()} \right)$
ω_ω^{CK}	$\left({}_{\left({}^{()} \right)} \right)$	$\psi_{\omega_1^{CK}}(\omega_\omega^{CK})$	$\left({}^{\left({}_{\left({}^{()} \right)} \right)} \right)$
$\omega_{\omega_1^{CK}}^{CK}$	$\left({}_{\left({}_{()} \right)} \right)$	$\psi_{\omega_2^{CK}}(\omega_{\omega_1^{CK}}^{CK})$	$\left(\begin{smallmatrix} \left({}_{\left({}_{()} \right)} \right) \\ () \end{smallmatrix} \right)$

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- **Theorem** (DFD, JjJ) For recursive Λ we have GLP_Λ sound and complete for the omega rule interpretation for a large class of theories

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- If $\phi \in \Sigma^0_{2m+1}(X, x)$ then
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- $\mathrm{ACA}_0 \vdash \mathtt{wo}(\Lambda) \to \Big(\forall\, \lambda {\in} |\Lambda|\ \big(\lambda > 0 \to$
  $[\lambda|\Lambda, X]^{\Lambda}_{\mathrm{ECA}_0} \mathtt{TI}^{\overline{\Lambda}}_{\omega\cdot\dot{\lambda}}(\phi(\overline{X}))\big)\Big)$.

Towards a Π^0_1 analysis of predicativity

Predicative oracle consistency:

$$\texttt{Pred-O-Con}(T) = \forall \Lambda \forall X (\texttt{wo}(\Lambda) \to \langle \Lambda | X \rangle_T \top)$$

Theorem (Cordón-Franco, DFD, JjJ, Lara-Martín)

$$\mathrm{ATR}_0 \equiv \mathrm{ECA}_0 + \texttt{Pred-O-Con}(\mathrm{ECA}_0)$$

- Recall: $\mathrm{PA} \equiv \mathrm{EA} + \{n\text{-}\texttt{Con}(\mathrm{EA}) \mid n < \omega\}$.
- $\mathrm{ATR}_0 \equiv \mathrm{ECA}_0 +$ "$\{\alpha\text{-}\texttt{Oracle-Con}(\mathrm{ECA}_0) \mid \alpha$ a well-order$\}$".

Conjectures:

- $\mathrm{ATR}_0 \equiv_{\Pi^0_1} \mathrm{EA} + \{\langle\gamma\rangle_{\mathrm{EA}}\top : \gamma < \Gamma_0\}$
- $\|\mathrm{ATR}_0\|^{\mathrm{ECA}_0}_{\Pi^0_1} = \Gamma_0$

$[\infty]_T\phi$ holds if ϕ is provable using an *arbitrary* number of ω-rules.

Theorem (DFD):

$$\Pi^1_1\text{-CA} = \mathrm{ECA}_0 + \forall X\ \langle\infty|X\rangle_{\mathrm{ECA}_0}\top$$

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- Friedman, Goldfarb and Harrington come to the rescue!

Theorem

Let T be any computably enumerable theory extending EA and let $n < \omega$. For each $\sigma \in \Sigma^0_{n+1}$ we have that there is some $\rho_n \in \Sigma^0_{n+1}$ so that

$$\mathrm{EA} \vdash \langle n \rangle^{\mathsf{True}}_T \top \to \ \big(\sigma \leftrightarrow [n]^{\mathsf{True}}_T \rho_n\big).$$

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- **proof** The proof runs analogue to the proof of the classical FGH theorem adding an additional ingredient to get things down to EA.

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- Theorem
The logic GLP_Λ *is sound for strong enough theories* T *under the interpretation* $\Box \mapsto [\lambda]^{\Box,\Lambda}_T$.

Definition

Let T be a c.e. theory. We define

- $\Delta_0^{\Box} := \Sigma_0^{\Box} := \Pi_0^{\Box} := \Delta_0^0$;
- $\Sigma_{\alpha+1}^{\Box} = \Sigma_\alpha^{\Box} \cup \Pi_\alpha^{\Box} \cup \{[\alpha]_T^{\Box}\varphi(\dot{x}) \mid \varphi(x) \in \mathsf{Form}\}$ for $\alpha > 0$;
- $\Pi_{\alpha+1}^{\Box} = \Sigma_\alpha^{\Box} \cup \Pi_\alpha^{\Box} \cup \{\langle\alpha\rangle_T^{\Box}\varphi(\dot{x}) \mid \varphi(x) \in \mathsf{Form}\}$ for $\alpha > 0$;
- $\Sigma_\lambda^{\Box} := \Pi_\lambda^{\Box} := \bigcup_{\alpha<\lambda} \Sigma_\alpha^{\Box}$ for $\lambda \in \mathsf{Lim}$.

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- No longer runs out of phase

Theorem

Let T be a c.e. theory containing $\mathtt{ECA}_0$.

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1. **Theorem/conjecture:** The theory $\mathrm{EA} + \{\langle\xi\rangle\top \mid \xi < \zeta\}$ has Π^0_1 ordinal $\sup\{e^{\xi}1 \mid \xi < \zeta\}$.
2. **Main question**: how do these theories relate to better known theories like fragments of second order arithmetic of weak set-theories.