Huiswerkopdrachten II, Week 3 & 4[*]

HW = huiswerkopdrachten.

Natuurlijke deduktie

1.
$$\dfrac{\dfrac{\dfrac{[p]_2 \quad [p \to \bot]_1}{\bot}\ {\to}E}{(p \to \bot) \to \bot}\ {\to}I_1}{p \to ((p \to \bot) \to \bot)}\ {\to}I_2$$

2.
$$\dfrac{\dfrac{\dfrac{\dfrac{[p]_2 \quad [\neg p]_1}{\bot}\ {\to}E}{q}\ \bot E}{\neg p \to q}\ {\to}I_1}{p \to (\neg p \to q)}\ {\to}I_2$$

3.
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[p]_2}{p \vee \neg p}\ \vee I, l \qquad [\neg(p \vee \neg p)]_1}{\bot}\ {\to}E}{\neg p}\ {\to}I_2}{p \vee \neg p}\ \vee I, r \qquad [\neg(p \vee \neg p)]_1}{\bot}\ {\to}E}{p \vee \neg p}\ RAA_1$$

4.
$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\neg p]_2}{\neg p \vee \neg\neg p}\ \vee I, l \qquad [\neg(\neg p \vee \neg\neg p)]_1}{\bot}\ {\to}E}{\neg\neg p}\ {\to}I_2}{\neg p \vee \neg\neg p}\ \vee I, r \qquad [\neg(\neg p \vee \neg\neg p)]_1}{\bot}\ {\to}E}{\neg p \vee \neg\neg p}\ RAA_1$$

5. Geen tautologie

6.
$$\dfrac{\dfrac{\dfrac{[p \vee q]_1 \qquad \dfrac{\dfrac{[p]_4 \quad [\neg p]_2}{\bot}\ {\to}E}{q}\ \bot E \qquad [q]_3}{q}\ \vee E_{3,4}}{\neg p \to q}\ {\to}I_2}{p \vee q \to (\neg p \to q)}\ {\to}I_1$$

[*] geLaTeXed door Ulrich

7.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[p]_4 \quad [p \to (q \vee r)]_1}{q \vee r} \to E \quad \dfrac{[q]_5 \quad [\neg q]_2}{\bot} \to E \quad \dfrac{\dfrac{[r]_6}{\neg p \vee r} \vee I, l \quad [\neg(\neg p \vee r)]_3}{\bot} \to E}{\bot} \vee E_{5,6}}{\neg p} \to I_4}{\neg p \vee r} \vee I, l \quad [\neg(\neg p \vee r)]_3}{\bot} \to E}{\neg p \vee r} \to RAA_3}{\neg q \to (\neg p \vee r)} \to I_2}{(p \to (q \vee r)) \to (\neg q \to (\neg p \vee r))} \to I_1$$

8. Geen tautologie

9. **HW**

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[p]_3 \quad [p \to q]_1}{q} \to E \quad [\neg q]_2}{\bot} \to E}{\neg p} \to I_3}{\neg q \to \neg p} \to I_2}{(p \to q) \to (\neg q \to \neg p)} \to I_1$$

10. **HW** Moet het niet RAA_4 zijn i.p.v. $\bot E_4$, omdat ik uit het niet verder kunnen deduceren vanuit q, $\neg q$ aanneem??

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[p]_5 \quad \dfrac{[\neg q]_4 \quad [\neg q \to \neg p]_6}{\neg p} \to E}{\bot} \to E}{q} \bot E_4}{p \to q} \to I_5}{(\neg q \to \neg p) \to (p \to q)} \to I_3 \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[p]_1 \quad [p \to q]_3}{q} \to E \quad [\neg q]_2}{\bot} \to E}{\neg p} \to I_1}{\neg q \to \neg p} \to I_2}{(p \to q) \to (\neg q \to \neg p)} \to I_6}{(p \to q) \leftrightarrow (\neg q \to \neg p)} \wedge I$$

11.

$$\dfrac{\dfrac{\dfrac{[p]_6 \quad \dfrac{[(p \wedge q) \vee (p \wedge r)]_7 \quad \dfrac{\dfrac{[p \wedge q]_4}{q} \wedge E, r}{q \vee r} \vee I, l \quad \dfrac{\dfrac{[p \wedge r]_5}{r} \wedge E, r}{q \vee r} \vee I, r}{q \vee r} \vee E_{4,5}}{p \wedge (q \vee r)} \wedge I_6}{(p \wedge q) \vee (p \wedge r) \to p \wedge (q \vee r)} \to I_7 \quad \dfrac{\dfrac{\dfrac{[p \wedge (q \vee r)]_3}{q \vee r} \wedge E, r \quad \dfrac{\dfrac{\dfrac{[p \wedge (q \vee r)]_3}{p} \wedge E, l \quad [q]_1}{p \wedge q} \wedge I \quad \dfrac{\dfrac{[p \wedge (q \vee r)]_3}{p} \wedge E, l \quad [r]_2}{p \wedge r} \wedge I}{(p \wedge q) \vee (p \wedge r)} \vee I}{(p \wedge q) \vee (p \wedge r)} \vee E_{1,2}}{p \wedge (q \vee r) \to (p \wedge q) \vee (p \wedge r)} \to I_3}{p \wedge (q \vee r) \leftrightarrow (p \wedge q) \vee (p \wedge r)} \wedge I$$

Konstrukties met bewijzen

1. Als $\vdash \varphi\wedge\psi$ dan $\vdash \varphi$ en $\vdash \psi$ hieruit volgt: $\overset{\mathcal{D}}{\varphi\wedge\psi}$ hieruit volgt: $\dfrac{\overset{\mathcal{D}}{\varphi\wedge\psi}}{\varphi}\ \wedge E, l$
   en $\dfrac{\overset{\mathcal{D}}{\varphi\wedge\psi}}{\psi}\ \wedge E, r$ hieruit volgt: $\vdash \varphi$ en $\vdash \psi$

2. Als $\vdash \varphi\vee\psi$ en $\vdash \neg\varphi$ dan $\vdash \psi$ hieruit volgt: $$\dfrac{\overset{\mathcal{D}}{\varphi\vee\psi} \qquad \dfrac{\dfrac{[\varphi]_1 \quad \overset{\mathcal{D}'}{\neg\varphi}}{\perp}\to E}{\psi}\perp \qquad [\psi]_2}{\psi}\ \vee E_{1,2}$$
   hieruit volgt: $\vdash \psi$

3. $\vdash \varphi$ en $\vdash \psi$ dan $\vdash \varphi\wedge\psi$ hieruit volgt: $\overset{\mathcal{D}}{\varphi}$ en $\overset{\mathcal{D}'}{\psi}$ hieruit volgt: $\dfrac{\overset{\mathcal{D}}{\varphi} \quad \overset{\mathcal{D}'}{\psi}}{\varphi\wedge\psi}\ \wedge I$
   hieruit volgt: $\vdash \varphi\wedge\psi$

4. **HW** $\vdash \varphi\to\psi$ en $\vdash \neg\psi$ dan ook $\vdash \neg\varphi$ hieruit volgt: $$\dfrac{\dfrac{\dfrac{[\varphi]_1 \quad \overset{\mathcal{D}}{\varphi\to\psi}}{\psi}\to E \qquad \overset{\mathcal{D}'}{\neg\psi}}{\perp}\to E}{\neg\varphi}\to I_1$$
   hieruit volgt: $\vdash \neg\varphi$

5. Als $\vdash \varphi\to\neg\varphi$ en $\vdash \neg\varphi\to\varphi$, dan $\vdash \perp$. $$\dfrac{\dfrac{\overset{\mathcal{D}}{\neg\varphi\to\varphi} \quad \neg\varphi}{\varphi}\to E \qquad \dfrac{\overset{\mathcal{D}'}{\varphi\to\neg\varphi} \quad \varphi}{\neg\varphi}\to E}{\perp}\to E$$

6. $\vdash \varphi\to\neg\varphi$ dan $\vdash \neg\varphi$ hieruit volgt: $$\dfrac{\dfrac{[\varphi]_1 \qquad \dfrac{[\varphi]_1 \quad \overset{\mathcal{D}}{\varphi\to\neg\varphi}}{\neg\varphi}\to E}{\perp}\to E}{\neg\varphi}\to I_1$$ hieruit volgt: $\vdash \neg\varphi$

Valuaties

1. (a) $v(\varphi\wedge\psi)$ iff $v(\varphi)\cdot v(\psi) = 1$
   (b) $v(\varphi\vee\psi)$ iff $v(\varphi)$ of $v(\psi) = 1$
   (c) $v(\varphi\vee\psi)$ iff $v(\varphi) + v(\psi) - v(\varphi)\cdot v(\psi)$
   (d) $v(\varphi\to\psi) = 0$ iff $v(\varphi) = 1$ en $v(\psi) = 0$
   (e) $v(\varphi\to\psi) = 0$ iff $1 - v(\varphi) + v(\varphi)\cdot v(\psi)$
   (f) $v(\neg\varphi) = 1$ iff $v(\varphi) = 0$
   (g) $v(\neg\varphi) = 1 - v(\varphi)$
   (h) $v(\varphi\leftrightarrow\psi) = 1$ iff $v(\varphi) = v(\psi)$
   (i) $v(\varphi\leftrightarrow\psi) = 1$ iff $1 - |v(\varphi) - v(\psi)|$

(j) $v(\perp) = 0$

2. Stel v zodanig dat $v(p) = v(q) = 1$ en $v(r) = 0$:

   (a) We willen inzien dat $v(\neg p \vee q \to q \wedge p) = 1$. Dit volgt uit $v(\neg p \vee q) = 1$ [want $v(q) = 1$], en $v(q \wedge p) = 1$ [omdat $v(q) = 1$ en $v(p) = 1$]. Ten slotte blijkt $v(\vee \to \wedge) = 1$ [omdat $v(1 \to 1) = 1$]

   (b) (en zo voorts . . .)

3. $\models \psi \Leftrightarrow \top \models \psi$, d.w.z. we willen bewijzen dat ψ waar is, omdat ψ waar is wanneer $\top$ waar is. Omdat $v(\top) = 1$ (per definitie), is $v(\psi) = 1$.

   In formuletaal: $\forall v\ [v(\top) = 1 \Rightarrow v(\psi) = 1]$

4. Moet nog $\ldots \subset$ (subset)

Semantiek en deduktie

1. We moeten laten zien dat $\forall v\ [v(\neg(p \to q)) = 1 \Rightarrow v(q \to p)) = 1]$ Oftewel: laten zien dat $(q \to p)$ waar is, gegeven dat $\neg(q \to p)$ waar is.

   Beschouw v zó dat $v(\neg(p \to q)) = 1$, dus $v(p) = 1$ en $v(q) = 0$. En v zó dat $v(q \to p)) = 1$, dus $v(p) = 1$ en $v(q) = 0$.

   In formuletaal: $v(\neg(q \to p)) = 1$ iff $v(q) = 1$ en $v(p) = 0$, $v(q \to p)) = 1$ iff $v(q) = 1$ en $v(p) = 0$

2 $\neg(p \to q) \vdash (q \to p)$

$$\cfrac{\cfrac{\cfrac{\cfrac{[q]_1}{p \to q}\ \to I \qquad \neg(p \to q)}{\perp}\ \to E}{p}\ \perp E}{q \to p}\ \to I_1$$

3 $\vdash \neg(p \to q) \to (q \to p)$

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{[q]_1}{p \to q}\ \to I \qquad [\neg(p \to q)]_2}{\perp}\ \to E}{p}\ \perp E}{q \to p}\ \to I_1}{\neg(p \to q) \to (q \to p)}\ \to I_2$$

4. $\models \neg(p \to q) \to (q \to p) = 1 \Rightarrow$

p	q	$p \to q$	$\neg(p \to q)$	$q \to p$	$\neg(p \to q) \to (q \to p)$
1	1	1	0	1	1
1	0	0	1	1	1
0	1	1	0	0	1
0	0	1	0	1	1

Tautologie.

5. We moeten laten zien dat $\forall v\ [v(p) = 1 \Rightarrow v(\neg\neg p) = 1]$ Beschouw v zó dat $v(p) = 1$. Dan is $v(\neg\neg p) = 1$

6. $p \vdash \neg\neg p$

$$\dfrac{\dfrac{p \quad [\neg p]_1}{\bot}\to E}{\neg\neg p}\to I_1$$

7. $\models p \to \neg\neg p \Rightarrow v(p \to \neg\neg p) = 1$ iff $v(p) = 1$ of $v(p) = 0$

8. $\vdash p \to \neg\neg p$

$$\dfrac{\dfrac{\dfrac{[p]_1 \quad [\neg p]_2}{\bot}\to E}{\neg\neg p}\to I_2}{p \to \neg\neg p}\to I_1$$

9. $\models (p \to q) \vee (q \to p)$

p	q	$p \to q$	$q \to p$	$(p \to q) \vee (q \to p)$
1	1	1	1	1
1	0	0	1	1
0	1	1	0	1
0	0	1	1	1

Tautologie.

$v((p \to q) \vee (q \to p)) = 1$ iff $v(p), v(q)\{0, 1\}$ **??**

10. $\vdash (p \to q) \vee (q \to p)$

$$\dfrac{\dfrac{\dfrac{\dfrac{[q]_3}{p \to q}\to I_2 \quad \dfrac{[p]_2}{q \to p}\to I_3}{(p \to q) \vee (q \to p)}\vee I \qquad [\neg((p \to q) \vee (q \to p))]_1}{\bot}\to E_2}{(p \to q) \vee (q \to p)}RAA_1$$

11. **HW**

We moeten laten zien dat $\forall v\ [v(\neg p \vee q) = 1 \Rightarrow v(p \to q) = 1]$. Beschouw v zó dat $v(\neg p \vee q) = 1$, dus $v(p) = 1$ en $v(q) = 1$, ook wanneer $v(p) = 0$ en $v(q) = 1$ of $v(q) = 0$.

Voor $v(p \to q) = 1$ geldt dat $v(p) = 1$ en $v(q) = 1$, danwel $v(p) = 0$ en $v(q) = 1$ of $v(q) = 0$.

12. **HW** $\neg p \vee q \vdash p \to q$:

$$\dfrac{\dfrac{\neg p \vee q \qquad \dfrac{\dfrac{[p]_1 \quad [\neg p]_3}{\bot}\to E}{q}\bot E \qquad [q]_2}{q}\vee E_{2,3}}{p \to q}\to I_1$$

13. **HW** $\models \neg p \vee q \to (p \to q) = 1 \Rightarrow v(\neg p \vee q \to (p \to q)) = 1 \Rightarrow$

p	q	$\neg p$	$\neg p \vee q$	$p \to q$	$\neg p \vee q \to (p \to q)$
1	1	0	1	1	1
1	0	0	0	0	1
0	1	1	1	1	1
0	0	1	1	1	1

Tautologie.

N.B. zo:

p	q	$\neg p \vee q$	$\to$	$p \to q$
1	1	1	**1**	1
1	0	0	**1**	0
0	1	1	**1**	1
0	0	1	**1**	1

wil Joost de waarheidstabel dus NIET!

14. **HW**

$$\dfrac{\dfrac{\dfrac{[\neg p \vee q]_4 \qquad \dfrac{\dfrac{[p]_1 \quad [\neg p]_3}{\bot}\,{\to}E}{q}\,\bot E \qquad [q]_2}{q}\,\vee E_{2,3}}{p \to q}\,{\to}I_1}{\neg p \vee q \to (p \to q)}\,{\to}I_4$$