Uitwerking Opgaves Voortgezette Logica Week I

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1 Natuurlijke Deductie

1.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\phi]^1 \quad [\psi]^2}{\phi \wedge \psi}\wedge I \qquad [\phi \wedge \psi \to \sigma]^3}{\sigma} \to E}{\psi \to \sigma} \to I, 2}{\phi \to (\psi \to \sigma)} \to I, 1}{(\phi \wedge \psi) \to (\phi \to (\psi \to \sigma))} \to I, 3$$

2.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\phi]^1 \quad [\psi]^2}{\phi \wedge \psi}\wedge I \qquad [\phi \wedge \psi \to \sigma]^3}{\sigma} \to E}{\phi \to \sigma} \to I, 1}{\psi \to (\phi \to \sigma)} \to I, 2}{(\phi \wedge \psi) \to (\psi \to (\phi \to \sigma))} \to I, 3$$

3.

$$\dfrac{\dfrac{\dfrac{\dfrac{[\phi \wedge \chi]^1}{\phi}\wedge E, l \qquad [\phi \to \psi]^2}{\psi} \to E}{\phi \wedge \chi \to \psi} \to I, 1}{(\phi \to \psi) \to (\phi \wedge \chi \to \psi)} \to I, 2$$

4.

$$\dfrac{\dfrac{[\phi \lor \psi]^1 \quad \dfrac{[\phi]^2}{\phi \lor (\phi \lor \psi)}\lor I, r \quad \dfrac{\dfrac{[\psi]^3}{\phi \lor \psi}\lor I, l}{\phi \lor (\phi \lor \psi)}\lor I, l}{\phi \lor (\phi \lor \psi)}\lor E, 2, 3}{(\phi \lor \psi) \to (\phi \lor (\phi \lor \psi))}\to I, 1$$

5.

$$\dfrac{\dfrac{[\phi \lor (\phi \lor \psi)]^1 \quad \dfrac{[\phi]^2}{\phi \lor \psi}\lor I, r \quad [\phi \lor \psi]^3}{\phi \lor \psi}\lor E, 2, 3}{\phi \lor (\phi \lor \psi) \to \phi \lor \psi}\to I, 1$$

6.

$$\dfrac{\dfrac{\dfrac{[\phi]^1 \quad [\psi]^2}{\phi \land \psi}\land I}{\psi \to \phi \land \psi}\to I, 2}{\phi \to (\psi \to \phi \land \psi)}\to I, 1$$

7.

$$\dfrac{\dfrac{\dfrac{[\phi \land \psi]^1}{\psi}\land E, r \quad \dfrac{[\phi \land \psi]^1}{\phi}\land E, l}{\psi \land \phi}\land I}{\phi \land \psi \to \psi \land \phi}\to I, 1$$

8.

$$\dfrac{\dfrac{[(\phi \land \psi) \lor \phi]^1 \quad \dfrac{[\phi \land \psi]^2}{\phi}\land E, l \quad [\phi]^3}{\phi}\lor E, 2, 3}{(\phi \land \psi) \lor \phi \to \phi}\to I, 1$$

9.

$$\dfrac{\dfrac{[\phi]^1}{(\phi \land \psi) \lor \phi}\lor I, l}{\phi \to (\phi \land \psi) \lor \phi}\to I, 1$$

10.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\phi]^1 \quad [\phi \to (\psi \land \chi)]^2}{\psi \land \chi}\to E}{\psi}\land E, l}{\phi \to \psi}\to I, 1 \qquad \dfrac{\dfrac{\dfrac{[\phi]^3 \quad [\phi \to (\psi \land \chi)]^2}{\psi \land \chi}\to E}{\chi}\land E, r}{\phi \to \chi}\to I, 3}{(\phi \to \psi) \land (\phi \to \chi)}\land I}{(\phi \to (\psi \land \chi)) \to (\phi \to \psi) \land (\phi \to \chi)}\to I, 2$$

11.

$$\dfrac{\dfrac{\dfrac{\dfrac{[\phi]^1 \quad \dfrac{[(\phi \to \psi) \land (\phi \to \chi)]^2}{\phi \to \psi}\land E}{\psi}\to E \qquad \dfrac{[\phi]^1 \quad \dfrac{[(\phi \to \psi) \land (\phi \to \chi)]^2}{\phi \to \chi}\land E}{\chi}\to E}{\psi \land \chi}\land I}{\phi \to \psi \land \chi}\to I, 1}{(\phi \to \psi) \land (\phi \to \psi) \to (\phi \to \psi \land \chi)}\to I, 2$$

12.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\phi]^1}{\phi \lor \psi}\lor I, r \quad [\phi \lor \psi \to \chi]^2}{\chi}\to E}{\phi \to \chi}\to I, 1 \qquad \dfrac{\dfrac{\dfrac{[\psi]^3}{\phi \lor \psi}\lor I, l \quad [\phi \lor \psi \to \chi]^2}{\chi}\to E}{\psi \to \chi}\to I, 3}{(\phi \to \chi) \land (\psi \to \chi)}\land I}{(\phi \lor \psi \to \chi) \to ((\phi \to \chi) \land (\psi \to \chi))}\to I, 2$$

13.

$$\dfrac{\dfrac{\dfrac{[(\phi \lor \psi) \land (\psi \to \phi)]^1}{\phi \lor \psi}\land E, l \qquad [\phi]^2 \qquad \dfrac{[\psi]^3 \quad \dfrac{[(\phi \lor \psi) \land (\psi \to \phi)]^1}{\psi \to \phi}\land E, r}{\phi}\to E}{\phi}\lor E, 2, 3}{(\phi \lor \psi) \land (\psi \to \phi) \to \phi}\to I, 1$$

14.

$$\dfrac{\dfrac{\dfrac{[\phi \lor \psi]^1 \qquad \dfrac{[\phi]^2 \quad \dfrac{[(\phi \to \chi) \land (\psi \to \chi)]^3}{\phi \to \chi}\land E, l}{\chi}\to E \qquad \dfrac{[\psi]^4 \quad \dfrac{[(\phi \to \chi) \land (\psi \to \chi)]^3}{\psi \to \chi}\land E, r}{\chi}\to E}{\chi}\lor E, 2, 4}{\phi \lor \psi \to \chi}\to I, 1}{(\phi \to \chi) \land (\psi \to \chi) \to (\phi \lor \psi \to \chi)}\to I, 3$$

2 Constructies met bewijzen

1. *Als* $\vdash \phi \wedge \psi$, *dan* $\vdash \phi$ *en* $\vdash \psi$
   Stel $\vdash \phi \wedge \psi$. Dan is er een derivatie
   $$\begin{array}{c} D \\ \phi \wedge \psi \end{array}$$
   In dat geval hebben we ook $\vdash \phi$:
   $$\dfrac{\begin{array}{c} D \\ \phi \wedge \psi \end{array}}{\phi} \wedge E, l$$
   En hebben we ook $\vdash \psi$:
   $$\dfrac{\begin{array}{c} D \\ \phi \wedge \psi \end{array}}{\psi} \wedge E, r$$
   Ofwel, gezien het feit dat de assumpties in D leeg was, kunnen we concluderen dat $\vdash \phi \wedge \psi$ impliceert $\vdash \phi$ en $\vdash \psi$.

2. *Als* $\vdash \phi \vee \psi$ *en* $\vdash \neg\phi$, *dan* $\vdash \psi$
   Gegeven dat $\vdash \phi \vee \psi$ en $\vdash \neg\phi$ hebben wij derivaties
   $$\begin{array}{c} D \\ \phi \vee \psi \end{array}$$
   en
   $$\begin{array}{c} D' \\ \neg\phi \end{array}$$
   Wij hebben dan tevens een derivatie
   $$\dfrac{\begin{array}{c} D \\ \phi \vee \psi \end{array} \quad \dfrac{\dfrac{[\phi]^1 \quad \begin{array}{c} D' \\ \neg\phi \end{array}}{\perp} \to E}{\psi} \perp \quad [\psi]^2}{\psi} \vee E, 1, 2$$
   Zodat, gezien het feit dat de assumpties in D en D' leeg waren, $\vdash \psi$. Ofwel: $\vdash \phi \vee \psi$ en $\vdash \neg\phi$ impliceert $\vdash \psi$.

3. *Als* $\vdash \phi$ *en* $\vdash \psi$, *dan* $\vdash \phi \wedge \psi$
   Gegeven dat $\vdash \phi$ en $\vdash \psi$ hebben wij derivaties
   $$\begin{array}{c} D \\ \phi \end{array}$$
   en
   $$\begin{array}{c} D' \\ \psi \end{array}$$
   Zodat tevens geldt $\vdash \phi \wedge \psi$ gegeven het feit dat de assumpties in D en D' leeg waren en gegeven de derivatie
   $$\dfrac{\begin{array}{c} D \\ \phi \end{array} \quad \begin{array}{c} D' \\ \psi \end{array}}{\phi \wedge \psi} \wedge I$$
   Ofwel: $\vdash \phi$ en $\vdash \psi$ impliceert $\vdash \phi \wedge \psi$.

4. *Als* $\vdash \phi \to \psi$ *en* $\vdash \neg\psi$, *dan* $\vdash \neg\phi$
   Gegeven $\vdash \phi \to \psi$ en $\vdash \neg\psi$ hebben wij derivaties

$$\begin{array}{c} D \\ \phi \to \psi \end{array}$$

en

$$\begin{array}{c} D' \\ \neg\psi \end{array}$$

Zodat tevens geldt $\vdash \neg\phi$, daar de assumpties in D en D' leeg waren en daar wij dan immers tevens hebben een derivatie

$$\cfrac{\cfrac{\cfrac{[\phi]^1 \qquad \overset{D}{\phi \to \psi}}{\psi}\,{\to}E \qquad \overset{D'}{\neg\psi}}{\bot}\,{\to}E}{\neg\phi}\,{\to}I,1$$

Ofwel, $\vdash \phi \to \psi$ en $\vdash \neg\psi$ impliceert $\vdash \neg\phi$.

5. *Als* $\vdash \phi \to \neg\phi$ *en* $\vdash \neg\phi \to \phi$, *dan* $\vdash \bot$
   Gegeven $\vdash \phi \to \neg\phi$ en $\vdash \neg\phi \to \phi$ hebben wij derivaties

   $$\begin{array}{c} D \\ \phi \to \neg\phi \end{array}$$

   en

   $$\begin{array}{c} D' \\ \neg\phi \to \phi \end{array}$$

   Wij hebben dan tevens een derivatie

   $$\cfrac{\cfrac{\cfrac{\cfrac{[\phi]^1 \qquad \cfrac{[\phi]^1 \qquad \overset{D}{\phi \to \neg\phi}}{\neg\phi}\,{\to}E}{\bot}\,{\to}E}{\neg\phi}\,{\to}I,1 \qquad \overset{D'}{\neg\phi \to \phi}}{\phi}\,{\to}E \qquad \cfrac{\cfrac{[\phi]^2 \qquad \cfrac{[\phi]^2 \qquad \overset{D}{\phi \to \neg\phi}}{\neg\phi}\,{\to}E}{\bot}\,{\to}E}{\neg\phi}\,{\to}I,2}{\bot}\,{\to}E$$

   Daar de vereniging van de sets van aannames in D en D' leeg is, geldt $\vdash \bot$.
   Ofwel: $\vdash \phi \to \neg\phi$ en $\vdash \neg\phi \to \phi$ impliceert $\vdash \bot$.

6. *Als* $\vdash \phi \to \neg\phi$, *dan* $\vdash \neg\phi$.
   Gegeven $\vdash \phi \to \neg\phi$ hebben wij een derivatie

   $$\begin{array}{c} D \\ \phi \to \neg\phi \end{array}$$

   Wij hebben dan tevens een derivatie

   $$\cfrac{\cfrac{[\phi]^1 \qquad \cfrac{[\phi]^1 \qquad \overset{D}{\phi \to \neg\phi}}{\neg\phi}\,{\to}E}{\bot}\,{\to}E}{\neg\phi}\,{\to}I,2$$

   Daar de set van aannames van D leeg is, geldt nu dus tevens $\vdash \neg\phi$. Conclusie: $\vdash \phi \to \neg\phi$ impliceert $\vdash \neg\phi$.

3 Valse honden en baasjes; Vertalingen

Opmerkingen vooraf: de term "`z'n`" in "Als een hond vals is, dan is `z'n` baas vals" interpreteer ik als "de". Geen slechte interpretatie, daar de term "`z'n`" lijkt te refereren naar een uniek figuur uit het discussiedomein. In termen van hogere-orde logica interpreteer ik "de" als

- $\lambda Y \lambda X \exists x(\forall y(Y(y) \leftrightarrow y = x) \wedge X(x))$

We hebben het weliswaar over hogere-orde logica, maar in de praktijk komt dit met beta-conversie neer op een eerste-orde logische term.

1. $\forall x(H(x) \to V(x))$
2. $\neg\exists x(H(x) \wedge V(x))$ $(\equiv \forall x(H(x) \to \neg V(x)))$
3. $\exists x(H(x) \wedge V(x))$
4. $\exists x(H(x) \wedge V(x)) \wedge \exists x(H(x) \wedge \neg V(x))$
5. Deze zin heeft twee lezingen m.b.t. scope-verhoudingen tussen de kwantoren (subject-wide lezing (1) is de meest voor de hand liggende, maar (2) bestaat ook):

   (1) $\forall x(H(x) \to \exists y(B(y, x)))$

   (2) $\exists x \forall y(H(y) \to B(x, y))$
6. Strikt gezien (in Montague grammatica) heeft deze zin ("sommige honden hebben een baasje") ook twee lezingen:

   (1) $\exists x(H(x) \wedge \exists y(B(y, x)))$

   (2) $\exists x \exists y(H(y) \wedge B(x, y))$
7. $\forall x(B(x, x))$
8. $\forall x((H(x) \wedge V(x)) \to \exists w(\forall y(B(y, x) \leftrightarrow y = w) \wedge V(w)))$
9. $\forall x((H(x) \wedge V(x)) \to \exists w(\forall y(B(y, x) \leftrightarrow y = w) \wedge H(w)))$
10. Deze zin heeft (logisch gezien) ten minste drie lezingen met betrekking tot het bereik van de negatie. In principe zijn er ook nog verschillende lezingen met betrekking tot scope verhoudingen tussen de kwantoren.

    (1) $\forall x(H(x) \to \exists w(\forall y(B(y, x) \leftrightarrow y = w) \wedge \neg H(w)))$

    (2) $\forall x(H(x) \to \neg\exists w(\forall y(B(y, x) \leftrightarrow y = w) \wedge H(w)))$

    (3) $\neg\forall x(H(x) \to \exists w(\forall y(B(y, x) \leftrightarrow y = w) \wedge H(w)))$
    $\equiv \exists x(H(x) \wedge \neg\exists w(\forall y(B(y, x) \leftrightarrow y = w) \wedge H(w)))$
11. $\exists x(H(x) \wedge \exists w(\forall y(B(y, x) \leftrightarrow y = w) \wedge B(x, y)))$
12. $\forall x(\exists y(B(y, x) \wedge V(y)) \to \exists y(H(y) \wedge B(x, y)))$

13. $\forall x(\exists y(B(y,x) \wedge V(y)) \rightarrow \exists y(H(y) \wedge V(y) \wedge B(x,y)))$

14. Verschillende lezingen m.b.t. scope van negatie. De tweede lezing lijkt in eerste instantie misschien onplausibel, maar in feite komt het erop neer dat er een specifiek baasje wordt genoemd, en niet zozeer bazen in het algemeen. Met deze ideeen lijkt de tweede lezing toch zeker aanwezig.

    (1) $\forall x(\exists y(B(x,y)) \rightarrow \neg H(x))$

    (2) $\neg\forall x(\exists y(B(x,y)) \rightarrow H(x))$
    $\equiv \exists x(\exists y(B(x,y)) \wedge \neg H(x))$

15. Twee lezingen. De tweede is semantisch niet plausibel, maar in dit geval is het toch zeker zeer duidelijk aanwezig.

    (1) $\forall x(\exists y(B(x,y)) \rightarrow H(x))$

    (2) $\exists x(H(x) \wedge \forall y(\exists w(B(y,w)) \rightarrow y = x))$

4 Predicaatlogica en natuurlijke deductie

1.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\forall x\phi(x)]^2}{\phi(y)}\,\forall E! \qquad \dfrac{[\forall x(\phi(x) \rightarrow \psi(x))]^1}{\phi(y) \rightarrow \psi(y)}\,\forall E!}{\psi(y)}\,\rightarrow E}{\forall x\psi(x)}\,\forall I!}{\forall x\phi(x) \rightarrow \forall x\psi(x)}\,\rightarrow I, 2}{\forall x(\phi(x) \rightarrow \psi(x)) \rightarrow (\forall x\phi(x) \rightarrow \forall x\psi(x))}\,\rightarrow I, 1$$

2.

$$\dfrac{\dfrac{\dfrac{\dfrac{[\forall x\phi(x)]^1}{\phi(y)}\,\forall E! \qquad \dfrac{[\forall x\neg\phi(x)]^2}{\neg\phi(y)}\,\forall E!}{\bot}\,\rightarrow E}{\neg\forall x\neg\phi(x)}\,\rightarrow I, 2}{\forall x\phi(x) \rightarrow \neg\forall x\neg\phi(x)}\,\rightarrow I, 1$$

3.

$$\dfrac{\dfrac{\dfrac{[\exists x\phi(x) \wedge \psi]^1}{\exists x\phi(x)}\,\wedge E, l \qquad \dfrac{\dfrac{[\phi(x)]^2 \qquad \dfrac{[\exists x\phi(x) \wedge \psi]^1}{\psi}\,\wedge E, r}{\phi(x) \wedge \psi}\,\wedge I}{\exists x(\phi(x) \wedge \psi)}\,\exists I!}{\exists x(\phi(x) \wedge \psi)}\,\exists E!, 2}{\exists\phi(x) \wedge \psi \rightarrow \exists x(\phi(x) \wedge \psi)}\,\rightarrow I, 1$$

Merk op dat de introductieregel van de kwantor is toegestaan daar $x \notin FV(\psi)$ (gegeven).

4.

$$
\dfrac{\dfrac{[\exists A(x) \vee \exists x B(x)]^1 \quad \dfrac{[\exists x A(x)]^2 \quad \dfrac{\dfrac{[A(x)]^3}{A(x) \vee B(x)}\,\vee I, r}{\exists x(A(x) \vee B(x))}\,\exists I!}{\exists x(A(x) \vee B(x))}\,\exists E!, 3 \quad \dfrac{[\exists x B(x)]^4 \quad \dfrac{\dfrac{[B(x)]^5}{A(x) \vee B(x)}\,\vee I, l}{\exists x(A(x) \vee B(x))}\,\exists I!}{\exists x(A(x) \vee B(x))}\,\exists E!, 5}{\exists x(A(x) \vee B(x))}\,\vee E, 2, 4}{\exists A(x) \vee \exists B(x) \to \exists x(A(x) \vee B(x))}\,\to I, 1
$$

5.

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\forall x(\phi(x) \vee \psi)]^1}{\phi(x) \vee \psi}\,\forall E! \quad [\phi(x)]^2 \quad \dfrac{\dfrac{\dfrac{[\psi]^4}{\forall x \phi(x) \vee \psi}\,\vee I, l \quad [\neg(\forall x \phi(x) \vee \psi)]^3}{\bot}\,\to E}{\phi(x)}\,\bot}{\phi(x)}\,\vee E, 2, 4}{\forall x \phi(x)}\,\forall I!}{\forall x \phi(x) \vee \psi}\,\vee I, r \quad [\neg(\forall x \phi(x) \vee \psi)]^3}{\bot}\,\to E}{\forall x \phi(x) \vee \psi}\,RAA, 3}{\forall x(\phi(x) \vee \psi) \to \forall x \phi(x) \vee \psi}\,\to I, 1
$$

6.

$$
\dfrac{\dfrac{[\forall x \phi(x) \vee \psi]^1 \quad \dfrac{\dfrac{\dfrac{[\forall x \phi(x)]^2}{\phi(x)}\,\forall E!}{\phi(x) \vee \psi}\,\vee I, r}{\forall x(\phi(x) \vee \psi)}\,\forall I! \quad \dfrac{\dfrac{[\psi]^3}{\phi(x) \vee \psi}\,\vee I, l}{\forall x(\phi(x) \vee \psi)}\,\forall I!}{\forall x(\phi(x) \vee \psi)}\,\vee E, 2, 3}{(\forall x \phi(x) \vee \psi) \to \forall x(\phi(x) \vee \psi)}\,\to I, 1
$$

De universele generalisatie over $\phi(x) \vee \psi$ bij premisse ψ is toegestaan daar $x \notin FV(\psi)$.

7.

$$\dfrac{\dfrac{\dfrac{[\exists x\neg\phi(x)]^1 \qquad \dfrac{\dfrac{[\forall x\phi(x)]^3}{\phi(x)}\,\forall E! \qquad [\neg\phi(x)]^2}{\bot}\rightarrow E}{\bot}\,\exists E!, 2}{\neg\exists x\neg\phi(x)}\rightarrow I, 1}{\forall x\phi(x)\rightarrow\neg\exists x\neg\phi(x)}\rightarrow I, 3$$

8.

$$\dfrac{\dfrac{\dfrac{[\exists\neg\phi(x)]^1 \qquad \dfrac{\dfrac{[\forall x\phi(x)]^3}{\phi(x)}\,\forall E! \qquad [\neg\phi(x)]^4}{\bot}\rightarrow E}{\bot}\,\exists E!, 4}{\neg\forall x\phi(x)}\rightarrow I, 3}{\exists x\neg\phi(x)\rightarrow\neg\forall x\phi(x)}\rightarrow I, 1$$

9.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\neg\phi(x)]^1}{\exists\neg\phi(x)}\,\exists I! \qquad [\neg\exists x\neg\phi(x)]^2}{\bot}\rightarrow E}{\phi(x)}\,RAA, 1}{\forall x\phi(x)}\,\forall I! \qquad [\neg\forall x\phi(x)]^5}{\bot}\rightarrow E}{\exists x\neg\phi(x)}\,RAA, 2}{\neg\forall x\phi(x)\rightarrow\exists\neg\phi(x)}\rightarrow I, 5$$

5 Constructieve Bewijzen

1.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\psi]^1 \quad [\neg\psi]^2}{\bot}\rightarrow E}{\neg\neg\psi}\rightarrow I, 2 \qquad [\neg\neg\neg\psi]^3}{\bot}\rightarrow E}{\neg\psi}\rightarrow I, 1}{\neg\neg\neg\psi\rightarrow\neg\psi}\rightarrow I, 3$$

Dit bewijs is constructief, daar geen gebruik is gemaakt van RAA.

2.

$$\dfrac{\dfrac{[\exists x(\phi(x) \vee \psi(x))]^1 \quad \dfrac{[\phi(x) \vee \psi(x)]^2 \quad \dfrac{\dfrac{[\phi(x)]^3}{\exists x\phi(x)}\;\exists I!}{\exists x\phi(x) \vee \exists x\psi(x)}\;\vee I, r \quad \dfrac{\dfrac{[\psi(x)]^4}{\exists x\psi(x)}\;\exists I!}{\exists x\phi(x) \vee \exists x\psi(x)}\;\vee I, l}{\exists x\phi(x) \vee \exists x\psi(x)}\;\vee E, 3, 4}{\exists x\phi(x) \vee \exists x\psi(x)}\;\exists E!, 2}{\exists x(\phi(x) \vee \psi(x)) \to \exists x\phi(x) \vee \exists x\psi(x)}\;\to I, 1$$

Dit bewijs is constructief, daar geen gebruik is gemaakt van RAA.

3.

$$\dfrac{\dfrac{[\exists x\phi(x) \vee \exists x\psi(x)]^1 \quad \dfrac{[\exists x\phi(x)]^2 \quad \dfrac{\dfrac{[\phi(x)]^3}{\phi(x) \vee \psi(x)}\;\vee I, r}{\exists x(\phi(x) \vee \psi(x))}\;\exists I!}{\exists x(\phi(x) \vee \psi(x))}\;\vee E, 3 \quad \dfrac{[\exists x\psi(x)]^4 \quad \dfrac{\dfrac{[\psi(x)]^5}{\phi(x) \vee \psi(x)}\;\vee I, l}{\exists x(\phi(x) \vee \psi(x))}\;\exists I!}{\exists x(\phi(x) \vee \psi(x))}\;\vee E, 5}{\exists x(\phi(x) \vee \psi(x))}\;\vee E, 2, 4}{\exists x\phi(x) \vee \exists x\psi(x) \to \exists x(\phi(x) \vee \psi(x))}\;\to I, 1$$

Dit bewijs is constructief, daar geen gebruik is gemaakt van RAA.

4.

$$\dfrac{\dfrac{\dfrac{[\exists(x)\phi(x)]^1 \quad \dfrac{[\phi(x)]^2 \quad \dfrac{[\forall x(\phi(x) \to \psi)]^3}{\phi(x) \to \psi}\;\forall E!}{\psi}\;\to E}{\psi}\;\exists E, 2}{\exists\phi(x) \to \psi}\;\to I, 1}{\forall x(\phi(x) \to \psi) \to (\exists x\phi(x) \to \psi)}\;\to I, 3$$

Dit bewijs is constructief, daar geen gebruik is gemaakt van RAA.

5.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{[\phi(x)]^1}{\exists x\phi(x)}\;\exists I! \quad [\exists x\phi(x) \to \psi]^2}{\psi}\;\to E}{\phi(x) \to \psi}\;\to I, 1}{\forall x(\phi(x) \to \psi)}\;\forall I!}{(\exists\phi(x) \to \psi) \to \forall x(\phi(x) \to \psi)}\;\to I, 2$$

Dit bewijs is constructief, daar geen gebruik is gemaakt van RAA.

6.

$$\dfrac{[\forall x(\phi(x) \to \psi]^1}{\forall x(\phi(x) \to \psi) \to \forall x(\phi(x) \to \psi)}\;\to I, 1$$

Dit bewijs is constructief, daar geen gebruik is gemaakt van RAA.