Recursion Theory

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- There is an interesting link from strange attractors in chaos theory to Goodstein's process.

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- Pure Predicate Calculus is undecidable
- Can be done directly by coding the halting problem, we give a shorter proof

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- Applications: dense linear ordering with no begin or end-points (Algebraicly closed fields of given characteristic)

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- The proof of (1) is a bit more involved

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- Thus we obtain that PC is creative!