Recursion Theory

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Talk by Terwijn

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- Final idea: $W_{f(x)} := W_e$ if $x \in K$

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- and $\varnothing$ otherwise.
- The case that $\varnothing$ has a code in A goes similar (misprint)

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- and much more

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- and denote it $\mathcal{D}_m$

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- use g^{-1} to get a set to apply f to, to obtain a new element. Use g again to get the element where is should be.

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- Well, if $W_{g(y)} = \{f(g(y))\}$!!!

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