Solutions for Exercise 6.2.9

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Exercise 6.2.9 . Let f be a natural number. Prove that there exist two c.e. sets B_0, B_1 such that

1. $B_0 \cup B_1 = W_f$,
2. $B_0 \cap B_1 = \emptyset$,
3. for any natural number e, if $W_e \searrow W_f$ is infinite, then $W_e \cap B_0 \neq \emptyset$ and $W_e \cap B_1 \neq \emptyset$.

Solution 1 (By Erica Neutel)**.**

Proof. Observe the following:

- $W_{f,s+1} - W_{f,s}$ is finite for any natural number s,
- $\{W_{f,s+1} - W_{f,s} \mid s \in \mathbb{N}\}$ is pairwise disjoint,
- $W_f = \bigcup_{s\in\mathbb{N}} (W_{f,s+1} - W_{f,s})$.

Construction

We will simultaneously construct $\langle B_0^s \mid s \in \mathbb{N}\rangle$, $\langle B_1^s \mid s \in \mathbb{N}\rangle$, and $\langle \chi_s \colon \{0, \cdots, s-1\} \to \{\text{available}, \text{unavailable}\} \mid s \in \mathbb{N}\rangle$ by induction on $s \in \mathbb{N}$.

At stage 0, $B_0^0 = B_1^0 = \emptyset$, χ_0 is the empty function.

At stage $s+1$, for $s \in \mathbb{N}$, we will add every $x \in W_{f,s+1} - W_{f,s}$ into either B_0^s or B_1^s by using χ_s.

For any $x \in W_{f,s+1} - W_{f,s}$, define $g(x)$ as follows:

$$g(x) = \mu e < s\,[x \in W_{e,s} \text{ and } \chi_s(e) = \text{ available.}]$$

- Note: rigorously, we have to revise B_0^{s+1}, B_1^{s+1}, χ_{s+1} at every time when we deal with each $x \in W_{f,s+1} - W_{f,s}$ separately. But we will skip that because I am lazy.

If $g(x)$ does not exist or if $g(x)$ exists and $W_{g(x),s} \cap B_0^s = \emptyset$, then $B_0^{s+1} = B_0^s \cup \{x\}$, $B_1^{s+1} = B_1^s$, $\chi_{s+1} = \chi_s \cup \{(s, \text{available})\}$.

If $g(x)$ exists and $W_{g(x),s} \cap B_0^s \neq \emptyset$, then $B_0^{s+1} = B_0^s$, $B_1^{s+1} = B_1^s \cup \{x\}$, $\chi_{s+1} = (\chi_s - \{(g(x), \text{available})\}) \cup \{(g(x), \text{unavailable}), (s, \text{available})\}$.

Set

$$B_0 = \bigcup_{s\in\mathbb{N}} B_0^s, \qquad B_1 = \bigcup_{s\in\mathbb{N}} B_1^s.$$

Verification

Since we give an effective algorithm to construct B_0, B_1, they are c.e. Also $B_0 \cap B_1 = \emptyset$ because we only add new elements to either B_0^s or B_1^s at any stage $s+1$. Moreover, by the above observation and construction of B_0 and B_1, $W_f = B_0 \cup B_1$. Finally we have to check 3. Given any natural number e such that $W_e \searrow W_f$ is infinite. We will show that $W_e \cap B_0 \neq \emptyset$ and $W_e \cap B_1 \neq \emptyset$.

First, note the following:

$$x \in W_e \searrow W_f \iff (\exists s \in \mathbb{N})\ x \in W_{e,s} \text{ and } x \in W_{f,s+1} - W_{f,s}.$$

Since there are only finitely many $e' < e$ and $W_e \searrow W_f$ is infinite, at some stage $s+1$, every $e' < e$ must be either unavailable or $W_{e',s} \cap (W_{f,s+1} - W_{f,s}) = \emptyset$, and there exists an $x \in W_{f,s+1} - W_{f,s}$ with $x \in W_{e,s}$. Hence $g(x) = e$ for such an x and $x \in B_0^{s+1}$ by the construction. Therefore, $W_e \cap B_0 \neq \emptyset$.

Again, since $W_e \searrow W_f$ is infinite, there exists $s' > s$ and there is an $x' \in W_{f,s'+1} - W_{f,s'}$ such that $x' \in W_{e,s'}$. By the definition of g, $g(x') = e$. Since we know that $W_{e,s'} \cap B_0^{s'} \neq \emptyset$, by the construction, $x \in B_1^{s'+1}$. Hence $W_e \cap B_1 \neq \emptyset$. ■

Solution 2 (From the book "Recursively enumerable sets and degrees" written by Robert Soare in 1987, Springer Verlag, page 181-182)**.**

Proof. Take an injective recursive function $h\colon \mathbb{N} \to \mathbb{N}$ such that $W_f = \text{range}(h)$. As I told you in the class, we modify the definition of $\langle W_{f,s} \mid s \in \mathbb{N}\rangle$ as follows:

$$W_{f,s} = \{h(0), \cdots, h(s-1)\}.$$

Even if we modify it as above, we can prove Exercise 5.2.18 by using the modified definition.

Construction

We will simultaneously construct $\langle B_0^s \mid s \in \mathbb{N}\rangle$ and $\langle B_1^s \mid s \in \mathbb{N}\rangle$ by induction on $s \in \mathbb{N}$.

At stage 0, $B_0^0 = B_1^0 = \emptyset$.

At stage $s+1$ for $s \in \mathbb{N}$, we will add $h(s)$ to either B_0^s or B_1^s. Define $g(s)$ as follows:

$$g(s) = \mu\langle e, i\rangle < s\ [h(s) \in W_{e,s} \text{ and } W_{e,s} \cap B_i^s = \emptyset].$$

(Since we only search such a pair $\langle e, i\rangle$ up to s, this computation halts.)

If $g(s)$ exists, then $B_i^{s+1} = B_i^s \cup \{h(s)\}$, $B_{1-i}^{s+1} = B_{1-i}^s$. Otherwise $B_0^{s+1} = B_0^s \cup \{h(s)\}$, $B_1^{s+1} = B_1^s$.

Set

$$B_0 = \bigcup_{s \in \mathbb{N}} B_0^s, \qquad B_1 = \bigcup_{s \in \mathbb{N}} B_1^s.$$

Verification

It is clear that B_0 and B_1 are c.e. Also since we took h as injective, $B_0 \cap B_1 = \emptyset$. Since we have added all $h(s)'s$, $B_0 \cup B_1 = W_f$. Finally we check 3. First, by the modification of the definition of $W_{f,s}$, the following holds:

$$x \in W_e \searrow W_f \iff (\exists s \in \mathbb{N})\ x = h(s) \in W_{e,s} \text{ for any } x \text{ and } e.$$

Take any e with $W_e \searrow W_f$ infinite and $i = 0$ or 1. Since there are only finitely many $\langle e', i' \rangle < \langle e, i \rangle$ and $W_e \searrow W_f$ is infinite, there exists an s such that $g(s) = \langle e, i \rangle$. Hence $h(s) \in B_i^{s+1} \cap W_{e,s}$. Therefore, $W_e \cap B_i \neq \emptyset$.

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