Modal Logic 2014–2015, (Code 569070)

Final exam

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Exercise 1

1. Suppose a normal modal logic L is given via a Hilbert-style axiomatization. Give the definition of local derivability ($\Gamma \vdash^{l}_{\mathsf{L}} \phi$) and of global derivability $\Gamma \vdash_{\mathsf{L}} \phi$.
2. Show that $\{\Diamond\top\}$ is locally GL consistent.
3. Show that $\{\Diamond\top\}$ is globally GL inconsistent.
4. State, without proof, what relation holds between local and global derivability.

Exercise 2

Recall that the proof system GK is the calculus in the language of basic modal logic that arises by adding the following rule to $\mathsf{G3}$:

$$\Box\,\frac{\Gamma \Longrightarrow A}{\Box\Gamma \Longrightarrow \Box A, \Delta}\ .$$

It is very important to note that the succedent of the antecedent of the $\Box$ rule consists of a single formula and not of a multi-set. To see this, let us consider the rule

$$\Box ?\,\frac{\Gamma \Longrightarrow \Delta}{\Box\Gamma \Longrightarrow \Box\Delta, \Delta'}\ .$$

where Δ is allowed to be a multi-set. Let GKL be the logic that is obtained by adding the $\Box$? rule to $\mathsf{G3}$.

1. Show that GKL is a proper extension of GK.

2. Provide a rough proof-strategy for proving cut elimination for GKL. For example, mention the main induction, the necessary inversion lemmata, etc. Note, you do not need to prove all the ingredients of the cut elimination proof in this exercise but you will need to just mention them and tell how they all relate to each other.

3. Give an axiomatization of GKL in Hilbert style and, assuming cut-elimination for GKL, prove the equivalence between the Hilbert style and Gentzen style formulation of GKL.

Exercise 3

1. Let $x_1 R x_2 \dots R x_{n+1}$ be a chain of possible worlds inside some model $\mathcal{M}$, where R is a transitive, and irreflexive accessibility relation. Let p be an arbitrary propositional variable. Show that $\Box p \to p$ can be false in at most one world x_i $(1 \leq i \leq n+1)$ in the above chain $x_1 R x_2 \dots R x_{n+1}$.

2. Prove that for any $n < \omega$, and any formulas $\varphi_1, \dots, \varphi_n$, we have that

$$\mathsf{GL} \vdash \Diamond^{n+2}\top \quad \to \quad \Diamond\Big(\bigwedge_{i=1}^{n} \Box\varphi_i \to \varphi_i\Big).$$

   (*Hint: use the previous item of this exercise.*)

Exercise 4

For some index set I, let $\{\mathcal{M}_i\}_{i\in I}$ be a collection of models with $\mathcal{M}_i := \langle M_i, R_i, V_i\rangle$. By $\oplus_{i\in I}\mathcal{M}_i$ we denote the disjoint union of the models $\mathcal{M}_i$. This is formally defined as follows. $\oplus_{i\in I}\mathcal{M}_i := \langle M, R, V\rangle$ where $M := \{\langle x, i\rangle \mid x \in M_i\}$ and $\langle x, i\rangle R \langle y, j\rangle \;:\Leftrightarrow\; i{=}j \;\wedge\; xR_i y$, and $\langle x, i\rangle \in V(p) \;:\Leftrightarrow\; x \in V_i(p)$.

Let $r \notin M$. We define $r\oplus_{i\in I}\mathcal{M}_i$ by putting r under any other world in $\oplus_{i\in I}\mathcal{M}_i$ and defining $r \notin V(p)$ for any propositional variable p. Thus, if we write $r\oplus_{i\in I}\mathcal{M}_i := \langle M', R', V'\rangle$, then $M' := M \cup \{r\}$, $R' := R \cup \{\langle r, x\rangle \mid x \in \oplus_{i\in I}\mathcal{M}_i\}$ and $V'(p) = V(p)$.

1. Let $\{\mathcal{M}_i\}_{i\in I}$ be a collection of models of GL (not necessarily frames of GL) and pick $r \notin \oplus_{i\in I}\mathcal{M}_i$. Prove that $r\oplus_{i\in I}\mathcal{M}_i$ is again a GL-model.

2. Use only the soundness theorem and the above to prove that $\{\Diamond\phi \mid \mathsf{GL} \nvdash \neg\phi\}$ is consistent.

3. Prove that the canonical model of GL has a reflexive point.

4. Prove that the canonical model of GL contains an uncountably large cluster of reflexive points that each stand in the R^{GL} relation to each other.