GLP Lecture 2: The logic of provability

Joost J. Joosten

Dept. Lògica, Història i Filosofia de la Ciència
Universitat de Barcelona

Thursday 15-12-2011
Barcelona

- Number theory can code syntax and reason about it.

- Number theory can code syntax and reason about it.
- If $n \in \mathbb{N}$, we denote by $\overline{n}$ its numeral $\overbrace{S \ldots S}^{n \text{ times}} 0$.

- Number theory can code syntax and reason about it.
- If $n \in \mathbb{N}$, we denote by $\overline{n}$ its numeral $\overbrace{S \ldots S}^{n \text{ times}} 0$.
- For a formula φ we denote by $\ulcorner \varphi \urcorner$ its Gödel number.

- Number theory can code syntax and reason about it.
- If $n \in \mathbb{N}$, we denote by $\overline{n}$ its numeral $\overbrace{S \ldots S}^{n \text{ times}} 0$.
- For a formula φ we denote by $\ulcorner \varphi \urcorner$ its Gödel number.
- We will not distinguish between numerals and their Gödel numbers.

- Number theory can code syntax and reason about it.
- If $n \in \mathbb{N}$, we denote by $\overline{n}$ its numeral $\overbrace{S \ldots S}^{n \text{ times}} 0$.
- For a formula φ we denote by $\ulcorner \varphi \urcorner$ its Gödel number.
- We will not distinguish between numerals and their Gödel numbers.
- In particular there are predicates like Formula(x) and $\text{Axiom}_{\text{PA}}(x)$.

- Number theory can code syntax and reason about it.
- If $n \in \mathbb{N}$, we denote by $\overline{n}$ its numeral $\overbrace{S \ldots S}^{n \text{ times}} 0$.
- For a formula φ we denote by $\ulcorner \varphi \urcorner$ its Gödel number.
- We will not distinguish between numerals and their Gödel numbers.
- In particular there are predicates like Formula(x) and $\mathrm{Axiom}_{\mathrm{PA}}(x)$.
- And a predicate $\mathrm{Proof}_T(p, x)$.

- $\mathrm{Bew}_{\mathrm{PA}}(x) := \exists p\, \mathrm{Proof}_{\mathrm{PA}}(p, x)$

- $\mathrm{Bew}_{\mathrm{PA}}(x) := \exists p \, \mathrm{Proof}_{\mathrm{PA}}(p, x)$
- Distributivity corresponds to formalized Modus Ponens

- $\mathrm{Bew}_{\mathrm{PA}}(x) := \exists p \, \mathrm{Proof}_{\mathrm{PA}}(p, x)$
- Distributivity corresponds to formalized Modus Ponens
- The **K4** axiom to *Provable* Σ_1*-completeness*

- Let $\mathrm{su}(x, y)$ denote the 'function' $\mathrm{su}(\ulcorner \varphi \urcorner, y) \mapsto \ulcorner \varphi(\overline{y}) \urcorner$

- Let $\text{su}(x, y)$ denote the 'function' $\text{su}(\ulcorner\varphi\urcorner, y) \mapsto \ulcorner\varphi(\overline{y})\urcorner$
- **Theorem** Let $\varphi(x)$ be a formula with free variable x. Then, there exists a sentence ψ such that

$$\psi \leftrightarrow \varphi(\ulcorner\psi\urcorner)$$

- Let $\mathrm{su}(x, y)$ denote the 'function' $\mathrm{su}(\ulcorner\varphi\urcorner, y) \mapsto \ulcorner\varphi(\overline{y})\urcorner$
- **Theorem** Let $\varphi(x)$ be a formula with free variable x. Then, there exists a sentence ψ such that

$$\psi \leftrightarrow \varphi(\ulcorner\psi\urcorner)$$

- **Proof**: consider $k = \mathrm{su}(x, x)$ and $\psi = \varphi(\mathrm{su}(k, k))$.

- Let $\mathrm{su}(x, y)$ denote the 'function' $\mathrm{su}(\ulcorner\varphi\urcorner, y) \mapsto \ulcorner\varphi(\overline{y})\urcorner$
- **Theorem** Let $\varphi(x)$ be a formula with free variable x. Then, there exists a sentence ψ such that

$$\psi \leftrightarrow \varphi(\ulcorner\psi\urcorner)$$

- **Proof**: consider $k = \mathrm{su}(x, x)$ and $\psi = \varphi(\mathrm{su}(k, k))$.
- Corollary: Tarski's undefinability of truth

- If this sentence is true then Santaclause exists

- If this sentence is true then Santaclause exists
- Consider the fixed-point:

$$\psi \leftrightarrow (\mathsf{Bew}_{\mathsf{PA}}(\ulcorner \psi \urcorner) \rightarrow A)$$

- If this sentence is true then Santaclause exists
- Consider the fixed-point:

$$\psi \leftrightarrow (\mathrm{Bew}_{\mathrm{PA}}(\ulcorner \psi \urcorner) \rightarrow A)$$

- Homework: Show that over **K**4 Löb's axiom is equivalent Löb's rule: $\frac{\Box A \rightarrow A}{A}$