GLP Lecture 1: Calibration of Proof-theoretical Strength

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- In meta-mathematical language:

$$\mathcal{F} \vdash \mathrm{Con}(\mathcal{R})$$

- where $\mathcal{F}$ is some undisputed part of mathematics consisting of finitary methods only, and $\mathcal{R}$ denotes *'real' mathematics*

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- However, partial realizations of Hilbert's programme have been obtained
- Most notably, Gentzen's consistency proof for Peano Arithmetic (1936)

- Peano Arithmetic (PA) is the formal arithmetical theory in the language $\{0, S, +, \cdot, 2^x\}$ axiomatized by the regular axioms for the constant and function symbols together with full induction:

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- Here PR-TI(ϵ_0) is transfinite induction up to ϵ_0 for primitive recursive (p.r.) predicates

$$\forall \alpha {\in} S\, [\forall \beta {\prec} \alpha\ A(\beta) \to A(\alpha)] \to \forall \alpha A(\alpha)$$

where S is some set on which $\prec$ defines a (p.r.) well-order of order type ϵ_0 and A is a p.r. predicate

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- It is tempting to conceive of $\text{PR-TI}(\epsilon_0)$ as the non-finitistic part encompassed by PA.
- And in analogy to this, one can define a norm that measures proof strengths for theories T as follows:

$$|T|_{\text{con}} := \min\{\alpha \mid \text{PRA} + \text{PR-TI}(\alpha) \vdash \text{Con(T)}\}$$

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  (where x being transitive means $\forall y{\in}x \forall z(z \in y \rightarrow z \in x)$, that is, each element y of x is also a subset of x)
- Ad (b.): There are *pathological* orderings known (Kreisel) such that ω would be $|T|_{\mathrm{Con}}$ for any T

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- Note that, as T is consistent, $\mathrm{OT}(\mathbb{N}, <_T) = \omega$.

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- There are some technical details here as well-foundedness is a Π^1_1 predicate and as such not definable in first-order theories.

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- Let S be a set of true Σ^1_1 sentences, then, under some fairly reasonable conditions

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- We can define the proof theoretic measure

$$|T|_{\mathrm{it}} := \min\{\alpha \mid \mathcal{F}_\alpha \vdash \mathrm{Con}(T)\}$$

where $\mathcal{F}$ is a suitably chosen finitistic fragment of arithmetic

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- However, provability logics yield two main advantages
  - All the calculations involved in determining $|T|_{\mathrm{it}}$ can be done within these logics
  - The logics suggest a very natural ordinal notation which is completely unambiguous up to the Feferman-Shütte ordinal Γ_0

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- In particular, the fragments $I\Sigma_n$ can be fully characterized in terms of consistency statements
- We need some notation and terminology to make this precise.

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- This fact is formalizable in EA whence for $\sigma \in \Sigma_1$

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- It is an easy theorem that $\mathrm{RFN}_{\Sigma_n}(T)$ is equivalent to Kleene's rule for Σ_n formulas:

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- All of the steps can be done within EA!

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- It is not hard to see that $\mathrm{RFN}_{\Sigma_{n+1}}(\mathrm{EA})\vdash\pi\rightarrow\langle n\rangle\pi$ for $\pi\in\Pi_{n+1}$ whence
- $\mathrm{RFN}_{\Sigma_{n+1}}(\mathrm{EA})\vdash\Pi_{n+1}{-}\mathrm{RR}^n(\mathrm{EA})$

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- Here EA^+ is the theory EA together with the axiom stating that super-exponentiation is a total function

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- This can be conceived as the proof theoretic ordinal of PA