History of Logic
Midterm Exam (Take Home)

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This Exam is a good indication as to the level of the final exam. Of course, we have only covered half of the material so far. Please submit your answers by email to me before May 23, say, 18:00. Of course, this week, you will not need to invent a question and answer two others, or do any of the discussion posts.

1. Enumerable and decidable sets.
   (a) Show that if A is computably enumerable and B is decidable, then $A \setminus B$ is computably enumerable.
   (b) Show that the intersection of two computably enumerable sets is again computably enumerable
   (c) It is not the case that A is computably enumerable and B is decidable, then $A \setminus B$ is decidable. Give a counterexample.
   (d) It is not the case that A is computably enumerable and B is computably enumerable, then $A \setminus B$ is computably enumerable. Give a counterexample.
2. With $\mathsf{RFN}_T(\Pi_1)$, we denote so-called Π_1-reflection. This says that any Π_1 statement (Goldbach-like) that is provable in T is actually true. In symbols:
   $$\Box_T \pi \quad \Rightarrow \quad \pi \qquad \text{for } \pi \in \Pi_1.$$
   Prove that $\mathsf{Con}(T)$ and $\mathsf{RFN}_T(\Pi_1)$ are equivalent.
3. Give an example of a consistent theory that proves its own inconsistency.
4. Show that Gödel's fixed point G is actually Goldbach-like.
5. Consider the Rosser Sentence from the book. Let us call it R_T for the moment.
   (a) Show that R_T is a Π_1 sentence.
   (b) Show that a theory T is consistent, then its Rosser sentence R_T will be undecidable in T.

6. Let us consider one half of the first incompleteness theorem where G stands for Gödel's fixed point.

(1)	$\vdash G \;\Rightarrow \vdash \mathsf{Bew}(\ulcorner G \urcorner)$	because of ...
(2)	$\vdash G \;\Rightarrow \vdash \neg\mathsf{Bew}(\ulcorner G \urcorner)$	because of ...
(3)	$\vdash G \;\Rightarrow \vdash \bot$	because of ...
(4)	$\mathsf{Con}(T) \;\Rightarrow \nvdash G$	because of ...

(a) Fill out the dots in the above reasoning giving a concise justification for the particular step.

(b) Show that Gödel's second incompleteness theorem implies the first.

(c) Show how a close inspection of the above reasoning, together with some extra observations and the fact that G is undecidable in T gives a proof of the second incompleteness theorem.