History of Logic
Final Exam (Take Home)

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Please submit your answers by email to me before June 28, say, 18:00. Those taking this exam for level 3 are expected to give more formal proofs.

1. In this exercise, PA stands for Peano Arithmetic as presented in the book.
   (a) Let $\pi \in \Pi^0_1$ be such that $\mathsf{PA} \nvdash \neg\pi$. Prove that then necessarily, π is true.
   (b) Give a PA proof to the effect that
   $$\forall x \, \forall y \; (x + y = y + x).$$
2. Prove that there is a model of true arithmetic, such that there are two non-standard elements c and d with $c < d$ such that $c + i \neq d$ for any standard number i.
3. Let f be a computable function and let T be a theory containing the by now so familiar sufficient amount of arithmetic. Explain informally why it is the case that if
   $$f(m) = n, \text{ then } T \vdash f(\overline{m}) = \overline{n}.$$
4. Recall Löb's theorem:
   $$T \vdash \mathsf{Bew}_T(\ulcorner A \urcorner) \to A \quad \Rightarrow \quad T \vdash A.$$
   (a) Prove Löb's theorem using Gödel's second incompleteness theorem as is sketched in the book.
   (b) Prove Gödel's second incompleteness theorem using Löb's theorem and a clever substitution.
   (c) Use Löb's theorem to prove[1] the formalized version of Löb's theorem:
   $$T \vdash \mathsf{Bew}_T(\ulcorner \mathsf{Bew}_T(\ulcorner A \urcorner) \to A \urcorner) \to \mathsf{Bew}_T(\ulcorner A \urcorner).$$

[1] Hint: take for A the formalization of Löb and use the fact that $T \vdash \mathsf{Bew}_T(\ulcorner B \to C \urcorner) \wedge \mathsf{Bew}_T(\ulcorner B \urcorner) \to \mathsf{Bew}_T(\ulcorner C \urcorner)$.

5. Consider the Rosser Sentence from the book. Let us call it R_T for the moment.

   (a) Show that R_T is a Π^0_1 sentence.

   (b) Suppose T is a theory such that it proves the *minimal number principle* for decidable predicates. That is, if φ is some decidable predicate, then
   $$T \vdash \exists x\ \varphi(x) \to \exists y\ (\varphi(y) \wedge \forall\, z{<}y\ \neg\varphi(z)).$$
   In the rest of this exercise, we shall assume that T is such a theory. Prove that,
   $$T \vdash \varphi \Rightarrow T \vdash \exists y\ (\mathsf{Proof}_T(y, \ulcorner\varphi\urcorner) \wedge \forall\, z{<}y \neg\mathsf{Proof}_T(z, \ulcorner\varphi\urcorner))$$

   (c) Show that if a theory T is consistent, then its Rosser sentence R_T will be undecidable in T.

6. The so-called *busy beaver* function B is defined as follows.
   $$B(w) = \max\{s \mid P_i(j) = s\ \&\ |\langle i, j\rangle| \leq w\}$$

   (a) Do you expect the busy beaver function to be a slow-growing or rather a fast growing function.

   (b) Use Chaitin's incompleteness theorem to prove that $K(s)$ is not a computable function.

   (c) Prove that the busy beaver function is not a computable function.

Choose one of the following two exercises.

A Prove that for any n, we can find a sequence of computable enumerable sets A_i such that
$$A_0 \subset A_1 \subset \ldots \subset A_n$$
and also such that $A_{i+1} \setminus A_i$ is not computably enumerable for $0 \leq i < n$.

B Write a mini-essay of at most 4 A-4 on one of the following topics:

- The incompleteness theorems and physics.
- The incompleteness theorems and the theory of the mind.
- The incompleteness theorems and the foundations of mathemtics.
- The incompleteness theorems, epistemology and truth.