

Decidability of some interpretability logics

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Interpretability logic

- ▶ Interpretability logics have a binary modal operator \triangleright .

- ▶ Basic interpretability logic **IL**:

classically valid formulas (in the new language, $\Box, \Diamond, \triangleright$);

$$\text{K } \Box(A \rightarrow B) \rightarrow (\Box A \rightarrow \Box B);$$

$$\text{Löb } \Box(\Box A \rightarrow A) \rightarrow \Box A;$$

$$\text{J1 } \Box(A \rightarrow B) \rightarrow A \triangleright B;$$

$$\text{J2 } (A \triangleright B) \wedge (B \triangleright C) \rightarrow A \triangleright C;$$

$$\text{J3 } (A \triangleright C) \wedge (B \triangleright C) \rightarrow A \vee B \triangleright C;$$

$$\text{J4 } A \triangleright B \rightarrow (\Diamond A \rightarrow \Diamond B);$$

$$\text{J5 } \Diamond A \triangleright A.$$

- ▶ rules: modus ponens and necessitation $A/\Box A$.

(parentheses priority: $\neg, \Box, \Diamond; \wedge, \vee; \triangleright; \rightarrow, \leftrightarrow$)

Models

- ▶ Semantics: extend the usual relational (Kripke) model.
- ▶ Veltman model: $\mathcal{M} = \langle W, R, \{S_w : w \in W\}, V \rangle$, where:
 1. $W \neq \emptyset$;
 2. R^{-1} is well-founded (no $x_0 R x_1 R x_2 R \dots$ chains);
 3. R is transitive;
 4. $S_w \subseteq R(w)^2$ is reflexive, transitive, contains $R \cap R(w)^2$ ($w R u R v$ implies $u S_w v$);
 5. $V : Prop \rightarrow \mathcal{P}(W)$.

- ▶ Truth of a formula $F \triangleright G$ (“ F interprets G ”) in a world $w \in \mathcal{M}$:

$$w \Vdash F \triangleright G \quad :\Leftrightarrow \quad \forall x \in R(w) : x \Vdash F \Rightarrow \exists y \in S_w(x) : y \Vdash G.$$

- ▶ **IL**-frame (Veltman frame) is a triple $\mathcal{F} = \langle W, R, \{S_w : w \in W\} \rangle$.
- ▶ We have:

$$\mathbf{IL} \vdash F \Leftrightarrow \forall \mathcal{F} : \mathcal{F} \vDash F.$$

Frame conditions

- ▶ Some extensions of **IL**:

$$\mathbf{ILM}_0 \quad \mathbf{IL} + A \triangleright B \rightarrow \diamond A \wedge \square C \triangleright B \wedge \square C$$

$$\mathbf{ILW} \quad \mathbf{IL} + A \triangleright B \rightarrow A \triangleright B \wedge \square \neg A$$

$$\mathbf{ILW}^* \quad \mathbf{IL} + A \triangleright B \rightarrow B \wedge \square C \triangleright B \wedge \square C \wedge \square \neg A$$

- ▶ $\mathbf{ILW}^* = \mathbf{ILM}_0 W \subseteq \mathbf{IL}(All)$
- ▶ These logics are complete w.r.t. certain classes of frames:

$$(M_0) \quad wRuRxS_wvRz \Rightarrow uRz;$$

$$(W) \quad S_w \circ R \text{ is reverse well-founded for each } w$$

$$(W^*) \quad (M_0) \text{ and } (W)$$

- ▶ **ILW**-frame is **IL**-frame that satisfies (W) etc.

Proving decidability

- ▶ Let's focus on **IL**.
- ▶ FMP: if $x \Vdash F$, then there is finite \mathcal{M} and $x' \in \mathcal{M}$ s.t. $x' \Vdash F$.
- ▶ Decision procedure: simultaneously do two things:
 - ▶ Enumerate the (countable) set of all **IL**-proofs.
 - ▶ Enumerate the (countable) set of (descriptions of) finite **IL**-models.
- ▶ The usual way of proving FMP is by filtrations.

Filtrations on **IL**-frames

- ▶ Let Γ contain A , closed under subformulas.
- ▶ Assume \sim is an equivalence relation on W , $\sim \subseteq \equiv_{\Gamma}$.
- ▶ For any $V \subseteq W$, define $\widetilde{V} = \{[v] \mid v \in V\}$.
- ▶ We define the rest of \widetilde{M} as follows.
- ▶ $\widetilde{R} = \{([w], [u]) \mid wRu, \exists \Box C \in \Gamma : w \not\models \Box C, u \Vdash \Box C\}$.
- ▶ $[u] \widetilde{S}_{[w]} [v]$ if and only if $[u], [v] \in \widetilde{R}([w])$, and for all/some $w' \in [w]$ and some $u' \in [u]$ such that $w'Ru'$ we have $u' S_{w'} v'$ for some $v' \sim v$.
- ▶ Define \Vdash so that x and $[x]$ agree on variables in Γ .
- ▶ We'll write R, S instead of $\widetilde{R}, \widetilde{S}$ when context allows.
- ▶ Problem: we lose transitivity of $S_{[w]}$.
 $w \rightarrow \{u \rightsquigarrow v_1 \sim v_2 \rightsquigarrow z\}, [w] \rightarrow \{[u] \rightsquigarrow [v] \rightsquigarrow [z]\}$

Filtrations on \mathbf{IL} -frames (2)

- ▶ Let Γ contain A , closed under subformulas.
- ▶ Assume \sim is an equivalence relation on W , $\sim \subseteq \equiv_{\Gamma}$.
- ▶ For any $V \subseteq W$, define $\widetilde{V} = \{[v] \mid v \in V\}$.
- ▶ We define the rest of \widetilde{M} as follows.
- ▶ $\widetilde{R} = \{([w], [u]) \mid wRu, \exists \Box C \in \Gamma : w \not\models \Box C, u \Vdash \Box C\}$.
- ▶ $[u] \widetilde{S}_{[w]} [v]$ if and only if $[u], [v] \in \widetilde{R}([w])$, and for some/all $w' \in [w]$ and all $u' \in [u]$ such that $w'Ru'$ we have $u' S_{w'} v'$ for some $v' \sim v$.
- ▶ Define \Vdash so that x and $[x]$ agree on variables in Γ .
- ▶ We'll write R, S instead of $\widetilde{R}, \widetilde{S}$ when context allows.
- ▶ Problem: we lose S_w -successors that don't agree enough.
 $w \rightarrow \{v_1[X] \leftarrow u_1 \sim u_2 \rightsquigarrow v_2[\neg X]\},$
 $[w] \rightarrow \{[u] \rightsquigarrow ? \}$

Generalized frames

- ▶ In the last example, ideally $[u] \rightsquigarrow \{v_1, v_2\}$.
- ▶ Generalized **IL**-frames (generalized Veltman frames).
- ▶ $\mathcal{M} = \langle W, R, \{S_w : w \in W\}, V \rangle$, where:
 1. $W \neq \emptyset$;
 2. R^{-1} is well-founded (no $x_0 R x_1 R x_2 R \dots$ chains);
 3. R is transitive;
 4. $S_w \subseteq R(w) \times 2^{R(w)}$ is:
 - ▶ non-empty;
 - ▶ quasi-reflexive $u S_w \{u\}$;
 - ▶ quasi-transitive $u S_w \{v_i \mid i \in I\}$ and $v_i S_w Z_i \Rightarrow u S_w \bigcup \{Z_i \mid i \in I\}$;
 - ▶ contains $R \cap R(w)^2$ $w R u R v$ implies $u S_w \{v\}$;
 - ▶ is monotonous $u S_w V \Rightarrow u S_w V', V \subseteq V'$
 5. $V : Prop \rightarrow \mathcal{P}(W)$.
- ▶ Truth of a formula $F \triangleright G$ (“ F interprets G ”) in a world $x \in \mathcal{M}$:
$$w \Vdash F \triangleright G \quad :\Leftrightarrow \quad \forall x \in R(w) : x \Vdash F \Rightarrow \exists V \in S_w(x) : V \Vdash G.$$
- ▶ $V \Vdash G$ stands for $v \Vdash G$ for all $v \in V$.

Filtration property

- ▶ $\widetilde{R} = \{([w], [u]) \mid wRu, \exists \Box C \in \Gamma : w \not\models \Box C, u \Vdash \Box C\}$.
- ▶ $[u] \widetilde{S}_{[w]} \widetilde{V}$ if and only if $\{[u]\}, \widetilde{V} \subseteq R([w])$, and for all $w' \in [w]$ and all $u' \in [u]$ such that $w' Ru'$ we have $u' S_{w'} V(w', u')$ for some $V(\widetilde{w}', u') \subseteq \widetilde{V}$.
- ▶ $w \rightarrow \{\{v_1[X]\} \leftarrow u_1 \sim u_2 \rightsquigarrow \{v_2[\neg X]\}\},$
 $[w] \rightarrow \{[u] \rightsquigarrow \{[v_1], [v_2]\}\}$
- ▶ Assume $\langle \widetilde{W}, \widetilde{R}, \widetilde{S}, \Vdash \rangle$ is a generalized frame (depends on \sim).
- ▶ Do we have $w \Vdash F \iff [w] \Vdash F$?

- ▶ Denote $[A]_w = \{x \in R[w] \mid x \Vdash A\}$.

Lemma

Let $w \Vdash A \triangleright B$. There is a maximal $u \in [A]_w$ such that

$$uS_w V \Rightarrow V \Vdash B.$$

We also have $u \Vdash \diamond A, B$.

Proof.

Existence: definition of \Vdash . Maximality: R is conversely well-founded. Since $uS_w\{u\}$, obviously $u \Vdash B$. Suppose $u \Vdash \diamond A$. Then $uRv \Vdash A$. Since $uS_w\{v\}$, by quasi-transitivity we have $S_w(v) \subseteq S_w(u)$. Contradiction with maximality of u . \square

Theorem

$$w \Vdash F \iff [w] \Vdash F.$$

Proof.

Induction on F .

\Leftarrow Assume $w \not\Vdash A \triangleright B$. Lemma: there is a maximal $u \in [A]_w$ such that $uS_w V \Rightarrow V \not\Vdash B$; and $u \not\Vdash \diamond A$.

We have $w \Vdash \diamond A$, and since $u \not\Vdash \diamond A$, $[w]R[u]$.

Let \widetilde{V} arbitrary s.t. $[u]S_{[w]}\widetilde{V}$. In particular, $uS_w V'$ for some $\widetilde{V}' \subseteq \widetilde{V}$. Since $V' \not\Vdash B$, by IH, $\widetilde{V}' \not\Vdash B$. Therefore $\widetilde{V} \not\Vdash B$.

□

Theorem

$$w \Vdash F \iff [w] \Vdash F.$$

Proof.

Induction on F .

\Rightarrow Assume $w \Vdash A \triangleright B$. Assume $[w]R[u] \Vdash A$. We construct \widetilde{V} s.t. $[w]R[u]S_{[w]}\widetilde{V} \Vdash B$.

Let $w' \in [w]$, $u' \in [u]$, wRu . Since $w' \sim w$, $w' \Vdash A \triangleright B$, therefore for some $V(w', u')$, $u'S_{w'}V(w', u') \Vdash B$.

For each point $v \in V(w', u')$, put $Z_v = \{v\}$ if $v \not\Vdash \Diamond B$.

Otherwise, $Z_v = \{m\}$, where m is arbitrary maximal world from $[B]_v$. Now, $vS_{w'}Z_v$, so by quasi-transitivity, $vS_{w'}\bigcup_v Z_v \Vdash \Box \neg B$.

Put $V := \bigcup_{w' \in [w], u' \in [u], wRu, v \in V(w', u')} Z_v$. By IH, $\widetilde{V} \Vdash B, \Box \neg B$.

It remains to show that $\widetilde{V} \subseteq R([w])$. This requires

$\exists C : \Box C \in \Gamma, [w] \not\Vdash \Box C, \widetilde{V} \Vdash \Box C$. Take $C = \neg B$.

□

- ▶ So, if $\langle \widetilde{W}, \widetilde{R}, \widetilde{S}, \Vdash \rangle$ is a model at all, then it is a filtration of $\mathcal{M} = \langle W, R, S, \Vdash \rangle$.
- ▶ Is it a model (does it satisfy quasi-transitivity etc.)? Depends on what \sim is.
- ▶ Ideally, x and $[x]$ are structurally similar, so that quasi-transitivity etc. is preserved.
- ▶ So, each $y \sim x$ should be structurally similar to x .

Definition

A bisimulation between **IL**-models $\langle W, R, \{S_w : w \in W\}, \Vdash \rangle$ and $\langle W', R', \{S'_{w'} : w' \in W'\}, \Vdash \rangle$ is any $Z \subseteq W \times W'$, $Z \neq \emptyset$:

(at) if wZw' then $w \Vdash p \iff w' \Vdash p$;

(forth) if wZw' and wRu , then there exists $u' \in R'(w')$ with uZu' and for all $V' \in S'_{w'}(u')$ there is $V \in S_w(u)$ such that for all $v \in V$ there is $v' \in V'$ with vZv' ;

(back) if wZw' and $w'R'u'$, then there exists $u \in R(w)$ such that uZu' and for all $V \in S_w(u)$ there is $V' \in S'_{w'}(u')$ such that for all $v' \in V'$ there is $v \in V$ with vZv' .

- ▶ By induction on F , if x and y are bisimilar (w.r.t. any bisimulation), $x \Vdash F \iff y \Vdash F$.
- ▶ Union of bisimulations (over generalized frames) is itself a bisimulation (*Vrgoč and Vuković, 2010*).
- ▶ In particular, there is a largest (auto)bisimulation $Z \subseteq W^2$.

- ▶ Denote by \sim the largest bisimulation on W^2 .
(equivalently, denote $x \sim y$ if there is any bisimulation at all which equates x and y)

Theorem

$\langle \widetilde{W}, \widetilde{R}, \widetilde{S}, \Vdash \rangle$ is a model.

Proof.

We should check: (1) $\widetilde{W} \neq \emptyset$, (2) \widetilde{R}^{-1} is well-founded, (3) \widetilde{R} is transitive, (4) $\widetilde{S}_{[w]} \subseteq \widetilde{R}([w]) \times 2^{\widetilde{R}([w])}$ (5) is quasi-reflexive $[u] \widetilde{S}_{[w]} \{[u]\}$, (6) quasi-transitive $[u] \widetilde{S}_{[w]} \{[v_i] \mid i \in I\}$ and $[v_i] \widetilde{S}_{[w]} Z_i \Rightarrow [u] \widetilde{S}_{[w]} \cup \{Z_i \mid i \in I\}$, (7) contains $\widetilde{R} \cap \widetilde{R}([w])^2$ $[w] \widetilde{R} [u] \widetilde{R} [v]$ implies $[u] \widetilde{S}_{[w]} \{[v]\}$, (8) is monotonous $[u] \widetilde{S}_{[w]} V \Rightarrow [u] \widetilde{S}_{[w]} V', V \subseteq V'$ □

Proof.

We should check: (1) $\tilde{W} \neq \emptyset$, (2) \tilde{R}^{-1} is well-founded, (3) \tilde{R} is transitive, (4) $\tilde{S}_{[w]} \subseteq \tilde{R}([w]) \times 2^{\tilde{R}([w])}$ (5) is quasi-reflexive $[u]\tilde{S}_{[w]}\{[u]\}$, (6) quasi-transitive $[u]\tilde{S}_{[w]}\{[v_i] \mid i \in I\}$ and $[v_i]\tilde{S}_{[w]}Z_i \Rightarrow [u]\tilde{S}_{[w]} \cup \{Z_i \mid i \in I\}$, (7) contains $\tilde{R} \cap \tilde{R}([w])^2$ $[w]\tilde{R}[u]\tilde{R}[v]$ implies $[u]\tilde{S}_{[w]}\{[v]\}$, (8) is monotonous $[u]\tilde{S}_{[w]}V \Rightarrow [u]\tilde{S}_{[w]}V', V \subseteq V'$.

Proof.

We should check: (1) $\widetilde{W} \neq \emptyset$, (2) \widetilde{R}^{-1} is well-founded, (3) \widetilde{R} is transitive, (4) $\widetilde{S}_{[w]} \subseteq \widetilde{R}([w]) \times 2^{\widetilde{R}([w])}$ (5) is quasi-reflexive

$[u]\widetilde{S}_{[w]}\{[u]\}$, (6) quasi-transitive $[u]\widetilde{S}_{[w]}\{[v_i] \mid i \in I\}$ and

$[v_i]\widetilde{S}_{[w]}Z_i \Rightarrow [u]\widetilde{S}_{[w]} \cup \{Z_i \mid i \in I\}$, (7) contains $\widetilde{R} \cap \widetilde{R}([w])^2$

$[w]\widetilde{R}[u]\widetilde{R}[v]$ implies $[u]\widetilde{S}_{[w]}\{[v]\}$, (8) is monotonous

$[u]\widetilde{S}_{[w]}V \Rightarrow [u]\widetilde{S}_{[w]}V', V \subseteq V'$.

(3). Assume $[w]R[u]R[v]$. Then (w.l.o.g.) $wRu \sim u'Rv$. Now (back) implies there is $v' \in R(u)$, $v' \sim v$. So $wRuRv'$, thus wRv' . Since $[w]R[u]$, there is A s.t. $w \Vdash \Box \neg A$, $u \Vdash \Box \neg A$. So, also $v' \Vdash \Box \neg A$. But then $[w]R[v']$. Since $v' \sim v$, $[w]R[v]$.

Proof.

We should check: (1) $\widetilde{W} \neq \emptyset$, (2) \widetilde{R}^{-1} is well-founded, (3) \widetilde{R} is transitive, (4) $\widetilde{S}_{[w]} \subseteq \widetilde{R}([w]) \times 2^{\widetilde{R}([w])}$ (5) is quasi-reflexive

$[u]\widetilde{S}_{[w]}\{[u]\}$, (6) quasi-transitive $[u]\widetilde{S}_{[w]}\{[v_i] \mid i \in I\}$ and

$[v_i]\widetilde{S}_{[w]}Z_i \Rightarrow [u]\widetilde{S}_{[w]} \cup \{Z_i \mid i \in I\}$, (7) contains $\widetilde{R} \cap \widetilde{R}([w])^2$

$[w]\widetilde{R}[u]\widetilde{R}[v]$ implies $[u]\widetilde{S}_{[w]}\{[v]\}$, (8) is monotonous

$[u]\widetilde{S}_{[w]}V \Rightarrow [u]\widetilde{S}_{[w]}V', V \subseteq V'$.

(3). Assume $[w]R[u]R[v]$. Then (w.l.o.g.) $wRu \sim u'Rv$. Now (back) implies there is $v' \in R(u)$, $v' \sim v$. So $wRuRv'$, thus wRv' . Since $[w]R[u]$, there is A s.t. $w \not\models \Box \neg A$, $u \Vdash \Box \neg A$. So, also $v' \Vdash \Box \neg A$. But then $[w]R[v']$. Since $v' \sim v$, $[w]R[v]$.

(7) Assume $[w]\widetilde{R}[u]\widetilde{R}[v]$. We already know $[w]\widetilde{R}[u]$ and $[w]\widetilde{R}[v]$.

Let $w' \sim w$, $u' \sim u$ such that $w'Ru'$. Since $u' \sim u$, (back) implies there is $v' \sim v$ such that $w'Ru'Rv'$. So for arbitrary $w' \sim w$, $u' \sim u$ there is v' s.t. $u'S_{w'}\{v'\}$ and indeed $[v'] \in \{[v]\}$. \square

- ▶ Thus, if \sim is the largest bisimulation on W^2 , then $\langle \widetilde{W}, \widetilde{R}, \widetilde{S}, \Vdash \rangle$ is a model, and a filtration.

We were trying to prove finite model property; is this a finite model?

- ▶ Each \widetilde{R} -transition eliminates at least one \diamond -formula from Γ ; so height is finite.
- ▶ Still, branching factor might be infinite.

Definition

A n -bisimulation between **IL**-models $\langle W, R, \{S_w : w \in W\}, \Vdash \rangle$ and $\langle W', R', \{S'_{w'} : w' \in W'\}, \Vdash \rangle$ is any sequence

$Z_n \subseteq \dots \subseteq Z_0 \subseteq W \times W'$:

(at) if wZ_0w' then $w \Vdash p \iff w' \Vdash p$;

(forth) if wZ_nw' and wRu , then there exists $u' \in R'(w')$ with $uZ_{n-1}u'$ and for all $V' \in S'_{w'}(u')$ there is $V \in S_w(u)$ such that for all $v \in V$ there is $v' \in V'$ with $vZ_{n-1}v'$;

(back) if wZ_nw' and $w'R'u'$, then there exists $u \in R(w)$ such that $uZ_{n-1}u'$ and for all $V \in S_w(u)$ there is $V' \in S'_{w'}(u')$ such that for all $v' \in V'$ there is $v \in V$ with $vZ_{n-1}v'$.

- ▶ Since height of \mathcal{M} is bounded by $|\Gamma|$, worlds are $|\Gamma|$ -bisimilar iff bisimilar.

- ▶ Put $u \equiv_n v$ if u and v agree on all formulas with at most n nested modalities.
- ▶ From now on, assume $Prop := Prop \cap \Gamma$.
- ▶ Now there are only finitely many formulas of modal depth up to n_0 (up to local equivalence).
- ▶ Denote $Th_n w$ the set of all formulas F s.t. $|F| \leq n$ and $w \Vdash F$.

Lemma

$$u \sim_n v \iff u \equiv_n v.$$

Proof.

\Rightarrow Induction on F .

\Leftarrow Induction on n . Step: assume (forth) doesn't hold.

Then there is $u \in R(w)$:

$$\begin{aligned} & (\forall u' \sim_{n-1} u, u' \in R(w')) (\exists V'(u') \in S_{w'}(u')) (\forall V \in S_w(u)) \\ & (\exists v(u', V) \in V) (\forall v' \in V'(u')) v(u', V) \not\sim_{n-1} v'. \end{aligned}$$

Put $B_V := \bigwedge_{u' \sim_{n-1} u, u' \in R(w')} Th_{n-1} v(u', V)$. Put

$B := \bigwedge_{V \in S_w(u)} \neg B_V$. For all $u' \sim u$, we have $V'(u') \Vdash B$
(because $v(u', V) \not\sim_{n-1} v'$).

Let $A := Th_{n-1} u$. Now $w' \Vdash A \triangleright B$. Since $w \equiv_n w'$, then
 $w \Vdash A \triangleright B$. Contradiction.



- ▶ Denote $\mathcal{N} = \widetilde{\mathcal{M}}$.
- ▶ For $x, y \in \mathcal{N}$, we now have $x \sim y \iff x \sim_{|\Gamma|} y \iff x \equiv_{|\Gamma|} y$.
- ▶ There are obviously only finitely many worlds in $\mathcal{M}/\equiv_{|\Gamma|}$.
- ▶ Since $\equiv_{|\Gamma|} = \sim_{|\Gamma|}$, $\widetilde{\mathcal{N}}$ (that is, $\widetilde{\widetilde{\mathcal{M}}}$) has only finitely many worlds.
- ▶ Thus we have FMP for **IL**.

Extending to **ILX**

- ▶ To prove FMP, given **ILX** that is complete w.r.t. class of Veltman frames that satisfy property C , we need to fill in the following:
 1. What is the (generalized) frame condition \mathcal{G} of X ?
 2. Is **ILX** complete w.r.t. to the class of \mathcal{G} -frames?
 3. Does $\widetilde{\mathcal{M}}$ have \mathcal{G} if \mathcal{M} has C ?
- ▶ For popular choices of X (except for W, W^*), 1 is known; and 2 usually reduces to completeness w.r.t. C .

Logic \mathbf{ILM}_0

- ▶ \mathbf{ILM}_0 is $\mathbf{IL} + A \triangleright B \rightarrow \diamond A \wedge \square C \triangleright B \wedge \square C$.
- ▶ Frame condition (VM):

$$wRuRxS_wvRz \Rightarrow uRz.$$

- ▶ Frame condition (GVM):

$$wRuRxS_wV \Rightarrow (\exists V' \subseteq V)(uS_wV' \ \& \ R(V') \subseteq R(u)).$$

- ▶ For each VM, there is a natural GVM (for xS_wy , $xS_w\{y\}$).
- ▶ Remains to prove $\widetilde{\mathcal{M}}$ preserves $(M_0)_{gen}$.

Theorem

If \mathcal{M} has property (M_0) , then $\widetilde{\mathcal{M}}$ has property (M_0) .

Proof.

Let $[w]R[u]R[x]S_{[w]}\widetilde{V}$. Fix $w' \in [w], u' \in [u]$. By bisimilarity, there is $x' \sim x, w'Ru'Rx'$.

Since $[x]S_{[w]}\widetilde{V}$, there is $V(w', u')$ such that $x'S_{w'}V(w', u')$ and $V(\widetilde{w'}, u') \subseteq \widetilde{V}$.

By (M_0) , there is $V'(w', u') \subseteq V(w', u')$ such that $R(V'(w', u')) \subseteq R(u')$.

Choose such $V'(w', u')$ for $w' \in [w], u' \in [u]$; $V' = \bigcup V'(w', u')$.

Now $[u]S_{[w]}\widetilde{V}$, remains to show $R(V') \subseteq R([u])$. Take $[v] \in V'$ and any $[z] \in R([v])$, w.l.o.g. we have vRz . By definition,

$v \sim v' \in V'(w', u')$ for some $v', w' \sim w, u' \sim u$. Since $v \sim v', v'Rz'$ for some $z' \sim z$. We had $R(V'(w', u')) \subseteq R(u')$.

So, $z' \in R(u')$. To show $[z] \in R([u])$, there should be a formula C , $[u] \Vdash \diamond C$, $[z] \not\Vdash \diamond C$. Take such C from $[v]R[z]$.

Since $v \sim v', v' \Vdash \diamond C$ and $R(V'(w', u')) \subseteq R(u')$, we have $u' \Vdash \diamond C$.

Logic ILW

▶ **ILW** is **IL** + $A \triangleright B \rightarrow A \triangleright B \wedge \Box \neg A$.

▶ Frame condition (VM):

$S_w \circ R$ is reverse well-founded for each w

▶ Frame condition (GVM)?

$$(\forall w \in W)(\forall X \subseteq R(w))(\forall Z \subseteq S_w^{-1}(X), Z \neq \emptyset)(\forall z \in Z) \\ (\exists V \subseteq X)(z S_w V \ \& \ (\forall v \in V)(R(v) \cap Z = \emptyset)).$$

▶ $(\forall Z \subseteq S_w^{-1}(X))$ is: for all Z such that for all $z \in Z$, $z S_w X$

▶ (Interestingly, equivalent after replacing $(\forall z \in Z)$ with $(\exists z \in Z)$; occasionally useful in proofs.)

Logic \mathbf{ILW}^*

- ▶ \mathbf{ILW}^* is $\mathbf{IL} + A \triangleright B \rightarrow B \wedge \Box C \triangleright B \wedge \Box C \wedge \Box \neg A$.
- ▶ $\mathbf{ILW}^* = \mathbf{ILWM}_0$.
- ▶ Frame condition (GVM)?
- ▶ Each \mathbf{ILW}^* -frame is \mathbf{ILW} -frame ($\mathbf{ILWM}_0 \supseteq \mathbf{ILW}$) and \mathbf{ILM}_0 -frame ($\mathbf{ILWM}_0 \supseteq \mathbf{ILM}_0$).
- ▶ Conversely, if \mathcal{F} is an both an \mathbf{ILW} -frame and an \mathbf{ILM}_0 -frame, then it is an \mathbf{ILWM}_0 -frame (induction on proof length).
- ▶ So, the frame condition is:

$$(W)_{gen} \text{ and } (M_0)_{gen}.$$

- ▶ If $\mathbf{ILW}^* \not\vdash F$, there is a \mathbf{ILM}_0 -, \mathbf{ILW} -VM \mathcal{M} , $w \in \mathcal{M}$, s.t. $w \not\Vdash F$.
Then $\widetilde{\mathcal{M}}$ is an \mathbf{ILM}_0 -, \mathbf{ILW} -model, and so an \mathbf{ILW}^* -model.

Complexity

- ▶ Given X , what is comp. complexity of $\{F \mid \mathbf{IL}X \vdash F\}$?
- ▶ Since $\mathbf{GL} \subseteq \mathbf{IL}$, at least PSPACE for any natural choice of X .
- ▶ The only (?) known result: \mathbf{IL}_0 is PSPACE-hard.
- ▶ Our goals:
 - ▶ \mathbf{IL} is in PSPACE;
 - ▶ \mathbf{ILW} is in PSPACE.
- ▶ (corollary: both are PSPACE-complete)

Complexity (2)

- ▶ Let F be any non-theorem of **ILX**. By completeness, there is \mathcal{M} , $w \in \mathcal{M}$ s.t. $w \not\models F$.
 1. Show that \mathcal{M} can be transformed to a certain model \mathcal{M}^f with some desirable properties:
 - ▶ accessibility relation (R) is a tree;
 - ▶ polynomial height;
 - ▶ polynomial branching factor;
 - ▶ S -relations should be “factorized”.
 2. Show that there is an algorithm that verifies the existence of all models with such properties.

Papers

-  T. Perkov, M. Vuković. Filtrations of generalized Veltman models. *Mathematical Logic Quarterly*, **62**, 412–419, 2016.
-  L. Mikec, T. Perkov, M. Vuković. Decidability of interpretability logics \mathbf{ILM}_0 and \mathbf{ILW}^* . *Submitted*.
-  (complexity paper - work in progress)