Introducció a la lògica 2015–2016, (Code 360906)

Practice second partial exam

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Exercise 0

We consider two sets A and B given by $A := \{\{0\}, 1, 2\}$ and $B := \{0, 1, \{2\}\}$. Compute the following:

1. $A \cap B =$
2. $A \cup B =$
3. $A \setminus B =$
4. $A \setminus B =$

Exercise 1

Give the definition of $\psi \models \varphi$ where ψ and φ are predicate logical sentences.

Exercise 2

What is the scheme of *reductio ad absurdum*?

Exercise 3

For each of the following statements, say if they are true or not. In case of logical consequence, give an argument, if we do not have logical consequence exhibit a counter-model.

1. $\forall x\, P(x) \to Qc \models \forall x\ (P(x) \to Qc)$;
2. $\forall x\ (P(x) \to Qc) \models \forall x\, P(x) \to Qc$;

3. $\exists x\, P(x) \to Qc \models \exists x\ (P(x) \to Qc)$;
4. $\exists x\ (P(x) \to Qc) \models \exists x\, P(x) \to Qc$;

Exercise 4

Using the language with only a binary relation R, give a finite collection Γ of sentences so that any model $\mathcal{A}$ that satisfies all sentences in Γ has to be infinite.

Exercise 5

Consider the predicate logical language with two predicates C and M, two names/constants t and j and one binary predicate R. We consider the following translation key:

Cx	x is a cat
Mx	x is a mouse
R x y	x chases y
t	Tom
j	Jerry

1. Translate the following sentences to our language of predicate logic using the given translation key.
   (a) Tom is a cat and Jerry is not a cat but a mouse;
   (b) Tom chases Jerry;
   (c) Any cat chases some mouse;
   (d) Some mouse chases any cat;
   (e) Any cat chases any mouse that does not chase any cat;
   (f) Any cat chases any mouse that does not chase some cat;
   (g) Any cat chases some mouse that does not chase some cat;
   (h) Some cat chases any mouse that does not chase some cat.
2. Of the following statements, say if they are true or not. In case of logical consequence, give an argument, if we do not have logical consequence exhibit a counter-model.
   (a) Any cat chases any mouse $\models$ Tom chases Jerry;
   (b) Any cat chases any mouse $\models$ some cat chases some mouse;
   (c) Any mouse is chased by any cat $\models$ any mouse is chased by some cat.

Exercise 6

We consider the language $\mathcal{L} := \{\overline{0}, S, \prec\}$ where $\overline{0}$ is a constant, and both S and $\prec$ are binary relations. Moreover, we consider two models $\mathcal{A}$ and $\mathcal{B}$ of $\mathcal{L}$ defined by:

$$\mathcal{A} := \langle \mathbb{N}, 0, \{\langle a, b\rangle \mid b = a + 1\}, \{\langle a, b\rangle \mid a < b\}\rangle$$

and

$$\mathcal{B} := \langle \mathbb{N} \cup \{\omega\}, 0, \{\langle a, b\rangle \mid b = a + 1\}, \{\langle a, b\rangle \mid a < b\} \cup \{\langle a, \omega\rangle \mid a \in \mathbb{N}\}.\rangle$$

Here, $\mathbb{N}$ stands for the set of natural numbers $\{0, 1, 2, 3, 4, \dots\}$ and $<$ and $+$ denote the regular less-than ordering and addition respectively on $\mathbb{N}$. Next, we consider $\mathcal{L}' := \mathcal{L} \cup \{P\}$ where P is a predicate. We consider the formulas

$$\varphi := \Big(P(\overline{0}) \wedge \forall x, y\ \big(\big(P(x) \wedge S(x, y)\big) \to P(y)\big)\Big) \to \forall z\ P(z);$$

and

$$\psi := \forall x\ \big(\forall y, x\ \big([\prec (y, x) \to P(y)] \to P(x)\big)\big) \to \forall z\ P(z).$$

1. Show that for any model $\mathcal{C} := \langle X, \overline{0}^{\mathcal{C}}, S^{\mathcal{C}}, \prec^{\mathcal{C}}, P^{\mathcal{C}}\rangle$ of $\mathcal{L}'$ we have that if $\langle X, \overline{0}^{\mathcal{C}}, S^{\mathcal{C}}, \prec^{\mathcal{C}}\rangle = \mathcal{A}$, then $\mathcal{C} \models \varphi$.

2. Show that for any model $\mathcal{C} := \langle X, \overline{0}^{\mathcal{C}}, S^{\mathcal{C}}, \prec^{\mathcal{C}}, P^{\mathcal{C}}\rangle$ of $\mathcal{L}'$ we have that if $\langle X, \overline{0}^{\mathcal{C}}, S^{\mathcal{C}}, \prec^{\mathcal{C}}\rangle = \mathcal{A}$, then $\mathcal{C} \models \psi$.

3. Show that for any model $\mathcal{C} := \langle X, \overline{0}^{\mathcal{C}}, S^{\mathcal{C}}, \prec^{\mathcal{C}}, P^{\mathcal{C}}\rangle$ of $\mathcal{L}'$ we have that if $\langle X, \overline{0}^{\mathcal{C}}, S^{\mathcal{C}}, \prec^{\mathcal{C}}\rangle = \mathcal{B}$, then $\mathcal{C} \models \psi$.

4. Show that there is some model $\mathcal{C} := \langle X, \overline{0}^{\mathcal{C}}, S^{\mathcal{C}}, \prec^{\mathcal{C}}, P^{\mathcal{C}}\rangle$ of $\mathcal{L}'$ so that $\langle X, \overline{0}^{\mathcal{C}}, S^{\mathcal{C}}, \prec^{\mathcal{C}}\rangle = \mathcal{B}$, and $\mathcal{C} \models \neg\varphi \wedge \psi$.