ON LAMBDA CALCULUS

JJJ

1. Our order of presentation

Since the book, in my modest opinion, is quite minimal in providing motivation we have tried to give extensive motivation in class. To better fit the narrative and rhetoric, we also changed the order of presentation.

(1) BHK treats implications as uniform constructions transforming any alleged proof of the antecedent into a proof of the conclusion.
(2) Thus, implication introduction can be denoted in lambda notation.
(3) Implication elimination corresponds to function application.
(4) Types and terms of the simply typed lambda calculus.
(5) Thus, formulas can be conceived as types and vice-versa.
(6) By induction, for each proof p in $\rightarrow \mathbf{N}m$ we can assign a term $T(p)$ where the free variable of $T(p)$ correspond to the labels of open assumptions in p and the conclusion of p corresponds to the type of $T(p)$.
(7) By induction, for each term t in $\mathbf{T}erm_{\rightarrow}$ we can assign a proof $P(t)$ in $\rightarrow \mathbf{N}m$ where the labels of the open assumptions of $P(t)$ correspond to the free variables of t.
(8) We can now state what is known as the Curry-Howard Isomorphism in a rudimentary form as the universal closures of $P(T(p)) = p$ and $T(P(t)) = t$.
(9) Sometimes states as Proofs stand to their conclusions as Terms to their types, as programs to their specifications.
(10) Lambda calculus is more than just a different notation system.
(11) To show this, we first define α conversion and substitution.
(12) We always assume that no free variables become bound after substitution and perform α conversion when needed.
(13) We show the Substitution Lemma and observe that it also holds for untyped Lambda calculus.
(14) With λ notation corresponding to function notation, both β and η reduction have natural justifications.
(15) We observe that in untyped lambda calculus $\Omega := (\lambda x.xx)(\lambda x.xx)$ reduces to itself.
(16) We define $\prec_{\beta,1}$ through a simple system.
(17) Which can be extended to define $\prec_{\beta}$ and $=_{\beta}$ as well as $\prec_{\beta,\eta}$ and $=_{\beta,\eta}$.
(18) We observe that β reduction on t corresponds to a natural transformation on $P(t)$.
(19) We observe that the Substitution Lemma gives rise to the substitutivity of $\prec_{\beta}$ and $\prec_{\beta,\eta}$.

(20) Rewrite systems come with natural notions like, normal forms, reduction sequences, reduction trees, strong and weak normalisation. Our leading example and motivation comes from conceiving elementary arithmetic on the naturals as a rewrite system.
(21) Likewise, we define confluence and weak confluence.
(22) Theorem 1.2.7 shows that confluence is good to have.
(23) Newman's Lemma tells us that weak confluence is also good and sometimes even implies confluence.
(24) We see that $\prec_{\beta,1}$ and $\prec_{\beta,\eta,1}$ are weakly confluent.
(25) We claim that moreover, they are strongly normalising not giving the proof.
(26) We do give a direct proof of confluence of $\prec_{\beta}$ and $\prec_{\beta,\eta}$.

2. Exercises

This is the first batch of exercises.

(1) Prove $\neg(A \wedge B) \to \neg A \vee \neg B$ in Natural Deduction.
(2) Prove $(A \to B) \vee (B \to A)$ in Natural Deduction.
(3) Prove $\neg\neg(\varphi \to \psi) \to (\neg\neg\varphi \to \neg\neg\psi)$ in Intuitionistic Natural Deduction.
(4) Give an example of an untyped lambda term that has a normal form but that is not strongly normalising.
(5) Give the inductive definition hinted at in Item 6 of the list in the previous section.
(6) Give the inductive definition hinted at in Item 7 of the list in the previous section.
(7) Prove the two equivalences mentioned in Item 8 of the list in the previous section. Hint: If you have done the previous exercises well, this should be a triviality.

This is the first batch of exercises. If you have them correct you get full score. Those who want can try the following

> **Bonus exercise** Strong normalisation of $\prec_{\beta}$ in the simply typed lambda calculus implies in particular that $\prec_{\beta,1}$ is irreflexive. Prove irreflexivity of $\prec_{\beta,1}$ on the set of simply typed lambda terms by elementary means.