AUTOMATED THEOREM PROVING

Final Exam

Exercise 1. By using Herbrand's Theorem, show that the formula

$$\varphi = \forall x \forall y \forall z (P(x, h(c), g(x, d)) \wedge \neg P(f(y), z, g(f(c), d)))$$

is unsatisfiable.

Solution: Put $\Phi = \{P(x, h(c), g(x, d)), \neg P(f(y), z, g(f(c), d))\}$. Clearly, $\varphi = \alpha_\Phi$. Let $\sigma = \{f(c)/x, c/y, h(c)/z\}$. Since $c, f(c)$ and $h(c)$ are elements of the Herbrand's universe of Φ, we have that $\Phi\sigma = \{P(f(c), h(c), g(f(c), d)), \neg P(f(c), h(c), g(f(c), d))\}$ is a set of ground instances of Φ. And obviously, $\Phi\sigma$ is unsatisfiable in the propositional sense. So, by Herbrand's theorem, $\varphi = \alpha_\Phi$ is unsatisfiable.

Exercise 2. Find all resolvents of the following two clauses:
$\varphi_1 = \neg P(x, y, u) \vee \neg P(y, z, v) \vee P(u, z, w) \vee Q(a, f(b))$,
$\varphi_2 = P(g(x, y), x, y) \vee \neg Q(x, x)$.

Solution: Since the variables are local in the clause in which they appear, we replace the variables x, y in φ_2 with variables x', y' respectively. We distinguish the following cases.

Case 1. $L = \{\neg P(x, y, u)\}$, $M = \{P(g(x, y), x, y)\}$ and $N = \{P(x, y, u), P(g(x', y'), x', y')\}$.

By using the unification algorithm, we see that N is unifiable by $\sigma_N = \{g(x', y')/x, x'/y, y'/u\}$. Hence, we obtain the resolvent

$$\neg P(x', z, v) \vee P(y', z, w) \vee Q(a, f(b)) \vee \neg Q(x', x').$$

Case 2. $L = \{\neg P(y, z, v)\}$, $M = \{P(g(x, y), x, y)\}$ and $N = \{P(y, z, v), P(g(x', y'), x', y')\}$.

By using the unification algorithm, we see that N is unifiable by $\sigma_N = \{g(x', y')/y, x'/z, y'/v\}$. Hence, we obtain the resolvent

$$\neg P(x, g(x', y'), u) \vee P(u, x', w) \vee Q(a, f(b)) \vee \neg Q(x', x').$$

Case 3. $L = \{\neg P(x, y, u), \neg P(y, z, v)\}$, $M = \{P(g(x, y), x, y)\}$ and $N = \{P(x, y, u), P(y, z, v), P(g(x', y'), x', y')\}$.

By using the unification algorithm, we see that N is not unifiable. For this, first we can match x with $g(x', y')$, and so we obtain

$$N_1 = N\{g(x', y')/x\} = \{P(g(x', y'), y, u), P(y, z, v), P(g(x', y'), x', y')\}.$$

Now, we match $g(x', y')$ with y, and thus we obtain

$$N_2 = N_1\{g(x', y')/y\} = \{P(g(x', y'), g(x', y'), u), P(g(x', y'), z, v), P(g(x', y'), x', y')\}.$$

But now, we can not match $g(x', y')$ with x'. So, in this case there is no resolvent

Case 4. $L = \{Q(a, f(b))\}$, $M = \{\neg Q(x, x)\}$ and $N = \{Q(a, f(b)), Q(x', x')\}$.

We have that N is not unifiable. For this, first we would match a with x', and thus we obtain $N_1 = N\{a/x'\} = \{Q(a, f(b)), Q(a, a)\}$. But now, we can not match $f(b)$ with a, and so there is no resolvent in this case.

Exercise 3. (1) Express the following facts by formulas in first-order logic:

(a) Every barber shaves all persons who do not shave themselves.

(b) No barber shaves any person who shaves himself.

For this, use $B(x)$ for "x is a barber", and $S(x, y)$ for "x shaves y".

(2) Prove by resolution that the conjunction of (a) and (b) implies that there are no barbers.

Solution: (1) We express fact (a) by the formula

$$\varphi_1 = \forall x(B(x) \to \forall y(\neg S(y, y) \to S(x, y))).$$

And we express fact (b) by the formula

$$\varphi_2 = \forall x \forall y((B(x) \wedge S(y,y)) \to \neg S(x,y)).$$

(2) We have to prove by resolution that the formula $\varphi_1 \wedge \varphi_2 \wedge \exists x B(x)$ is unsatisfiable.

We have that $\varphi_1 = \forall x(B(x) \to \forall y(\neg S(y,y) \to S(x,y))) \equiv \forall x(\neg B(x) \vee \forall y((S(y,y) \vee S(x,y))) \equiv \forall x \forall y(\neg B(x) \vee S(y,y) \vee S(x,y))$.

On the other hand, we have $\varphi_2 = \forall x \forall y((B(x) \wedge S(y,y)) \to \neg S(x,y)) \equiv \forall x \forall y(\neg(B(x) \wedge S(y,y)) \vee \neg S(x,y)) \equiv \forall x \forall y(\neg B(x) \vee \neg S(y,y) \vee \neg S(x,y))$.

Also, we take $B(c)$ as a Skolem standard form of $\exists x B(x)$. Then, we have the following proof by resolution:

1)	$\neg B(x) \vee S(y,y) \vee S(x,y)$	input
2)	$\neg B(x) \vee \neg S(y,y) \vee \neg S(x,y)$	input
3)	$B(c)$	input
4)	$S(y,y) \vee S(c,y)$	(1,3)
5)	$\neg S(y,y) \vee \neg S(c,y)$	(2,3)
6)	$\square$	(4,5)

In order to we obtain $\square$ in the last step of the resolution proof, we take $L = \{S(y,y), S(c,y)\}$, $M = \{\neg S(y,y), \neg S(c,y)\}$ and $N = \{S(y,y), S(c,y), S(y',y'), S(c,y')\}$. Since N is unifiable by $\sigma_N = \{c/y, c/y'\}$, we obtain the empty clause as a resolvent of the clauses $S(y,y) \vee S(c,y)$ and $\neg S(y,y) \vee \neg S(c,y)$.

Exercise 4. Ackermann's function is defined for every pair of natural numbers by means of the following equations:

$a(0,y) = y + 1$,
$a(x,0) = a(x-1,1)$ for $x > 0$,
$a(x,y) = a(x-1, a(x,y-1))$ for $x, y > 0$.

It is known that Ackermann's function is an example of a recursive function that is not primitive recursive. Then, write a Prolog program to compute Ackermann's function.

Solution:

$\text{ackermann}(0, Y, Z) :- Z \text{ is } Y + 1.$

$\text{ackermann}(X, 0, Z) :- X > 0,\ X1 \text{ is } X - 1,\ \text{ackermann}(X1, 1, Z).$

$$\text{ackermann}(X, Y, Z) :- X > 0,\ Y > 0,\ X1 \text{ is } X - 1,\ Y1 \text{ is } Y - 1, \\ \text{ackermann}(X, Y1, Z1), \text{ackermann}(X1, Z1, Z).$$

<u>Exercise 5</u>. (a) Write a Prolog program for the predicate union(L1,L2,L3), which means that L3 is the union of the lists L1 and L2.

(b) Write a Prolog program for the predicate intersection(L1,L2,L3), which means that L3 is the intersection of the lists L1 and L2.

Solution: (a)

$\text{union}([\,], L, L).$

$\text{union}([X|L1], L2, L3) :- \ \text{member}(X, L2), !, \text{union}(L1, L2, L3).$

$\text{union}([X|L1], L2, [X|L3]) :- \text{union}(L1, L2, L3).$

(b)

$\text{intersection}([\,], \_, [\,]).$

$\text{intersection}([X|L1], L2, [X|L3]) :- \ \text{member}(X, L2), !, \text{intersection}(L1, L2, L3).$

$\text{intersection}([\_|L1], L2, L3) :- \text{intersection}(L1, L2, L3).$